ANALYTIC R-GROUPS OF AFFINE HECKE ALGEBRAS

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1. **Introduction**

Let $\mathcal{H}$ be an affine Hecke algebra in the sense of Lusztig (see [16] or [25]). The Fréchet algebra $\mathcal{S}$, the Schwartz algebra completion of $\mathcal{H}$, is a central object of study for the harmonic analysis of $\mathcal{H}$. The key to understanding the structure of $\mathcal{S}$ is the Fourier transform isomorphism $\mathcal{F}$ which (by the main result of [9, Theorem 5.3]) identifies $\mathcal{S}$ with the algebra of Weyl group invariant sections of a certain smooth endomorphism bundle over the space Ξ_u of "tempered standard induction data."

Such a tempered standard induction datum consists of a triple (P, δ, t) where P denotes a subset of the set of simple roots of the based root system underlying $\mathcal{H}$. This subset P

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defines a "standard Levi subalgebra" $\mathcal{H}^P$ of $\mathcal{H}$, with semisimple quotient $\mathcal{H}_P$, and δ denotes a discrete series representation of $\mathcal{H}_P$. Finally t is a unitary induction parameter for $\mathcal{H}_P$, which is used to lift δ to a unitary representation δ_t of $\mathcal{H}^P$. The set of such unitary parameters has the structure of a compact real torus. Thus Ξ_u is a finite union of compact tori.

The endomorphism bundle alluded to above is constructed from a canonical projective unitary representation π of the groupoid $\mathcal{W}_{\Xi_u}$ of tempered standard induction data (cf. [9, Section 3.5]). The arrows in this groupoid are twists by certain isomorphisms of the $\mathcal{H}_P$ defined in terms of the (affine) Weyl group. This groupoid $\mathcal{W}_{\Xi_u}$ has been determined explicitly in general if $\mathcal{H}$ is of simple type, see [26]. The projective representation π yields a 2-cocycle $\gamma \in Z^2(\mathcal{W}_{\Xi_u}, \mathrm{U}(1))$ of $\mathcal{W}_{\Xi_u}$ with values in U(1).

The results [9, Theorem 3.11, Theorem 3.19] state that for every $\xi \in \Xi_u$ the representation $\pi(\xi)$ of $\mathcal{H}$ is unitary and tempered, and that for every irreducible tempered module ρ of $\mathcal{H}$ there exists a unique orbit $\mathcal{W}\xi$ (with $\xi \in \Xi_u$) such that ρ is equivalent to an irreducible summand of $\pi(\xi)$.

The main purpose of this article is to decompose $\pi(\xi)$ for $\xi \in \Xi_u$. For this reason we introduce a certain subgroup $\mathfrak{R}_\xi$, called the R-group, of the isotropy subgroup $\mathcal{W}_{\xi,\xi}$ of ξ in the Weyl groupoid $\mathcal{W}$. We show an analog of the Knapp-Stein Dimension Theorem [14], [30] which says that the commutant of the representation $\pi(\xi)$ has a basis given by the $\pi(\mathfrak{r})$ with $\mathfrak{r}$ running over the elements of $\mathfrak{R}_\xi$. As in [1, Section 2] it follows that there exists a bijective correspondence between the irreducible representations of $\mathcal{H}$ arising as summands of $\pi(\xi)$ on the one hand, and the irreducible representations of the twisted complex group algebra ${}^{\gamma_\xi}\mathbb{C}[\mathfrak{R}_\xi]$ on the other hand (where γ_ξ denotes the restriction of γ to $\mathfrak{R}_\xi$). This is the content of our main result, Theorem 5.5. If $\mathcal{H}$ is of simple type, our results yield the classification of the (equivalence classes of) irreducible tempered modules of $\mathcal{H}$. In the "equal parameter case" [13] and more generally for the affine Hecke algebras arising in the context of "unipotent representations" of inner forms of simple adjoint split groups, a classification of the tempered irreducible modules in terms of geometric data is also known [18].

The analogy with the theory of tempered representations of the group G of points of a reductive group defined over a local field is not surprising, since for various specializations of the parameters of $\mathcal{H}$ it is known that its module category is equivalent to a Bernstein block $\mathfrak{b}$ in the category of smooth representations of such a group G ([17], [21], [22], [23], [11]). If $\mathcal{H}$ is in fact isomorphic to the Hecke algebra of a "$\mathfrak{b}$-type" then this equivalence is known to respect temperedness and Plancherel measures [3]. Hence in this context our results yield the classification of the irreducible tempered representations of G which belong to $\mathfrak{b}$.

It is a fundamental question how the R-groups of parabolic induction for $\mathcal{H}$ which we will define below are related to the R-groups of parabolic induction for G if $\mathcal{H}$ is the Hecke algebra of a type for $\mathfrak{b}$ or in the context of [11]. Some general results in this direction have been achieved by Roche [29]

The R-groups and 2-cohomology classes $[\gamma_\xi]$ for classical affine Hecke algebras are amenable to *direct* explicit computation. This is illustrated by Slooten's computation [31] of the R-groups for classical Hecke algebras when the inducing representation is discrete series with real infinitesimal character (in the sense of [2]), and by the results in Section 6 of the present paper, proving the triviality of the 2-cocyles γ_ξ for classical Hecke algebras in all cases. With these results at hand, our decomposition theorem amounts in these cases to the proof of Slooten's conjectural classification [31, Conjecture 4.3(i)] of the irreducible tempered representations with real central character for classical Hecke algebras (see [31]). For a *geometric* approach to these results, see [7].

The main technical thrust of the proof of Theorem 5.5 is the fact that the $\mathcal{W}$-average of the product of a smooth section of the endomorphism bundle with the c-function is itself a smooth section (see equation (4.2)). Another technical tool is the computation of the constant term for generic parameters [9, Section 6.2].

2. **Affine Hecke algebras**

The structure of an affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is determined by an affine root datum (with basis) $\mathcal{R}$ together with a label function q defined on the extended affine Weyl group W associated to $\mathcal{R}$. We refer the reader to [16],[25],[9] for the details of the definition of the algebra $\mathcal{H}(\mathcal{R}, q)$, which we will only briefly review here.

Let $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ be a root datum (with basis $F_0 \subset X$ of simple roots of $R_0 \subset X$). Let W_0 denote the Weyl group of the reduced integral root system R_0. The extended affine Weyl group W associated with $\mathcal{R}$ is by definition $W = W_0 \ltimes X$. The affine root system R is equal to $R := R_0^\vee \times \mathbb{Z} \subset Y \times \mathbb{Z}$. Observe that R is closed for the natural action of W on the set of integral affine linear functions $Y \times \mathbb{Z}$ on X. Furthermore R is the disjoint union of the positive and the negative affine roots $R = R_+ \cup R_-$ as usual, and we define the length function l on W by

$$l(w) := |R_+ \cap w^{-1} R_-|. \tag{2.1}$$

The affine simple roots are denoted by F^{aff}.

A label function $q : W \to \mathbb{R}_+$ is a function which is length multiplicative (i.e. $q(uv) = q(u)q(v)$ if $l(uv) = l(u) + l(v)$) and which in addition satisfies $q(\omega) = 1$ if $l(\omega) = 0$. Thus a label function is completely determined by its values on the set S^{aff} of affine simple reflections in W. Observe that this gives rise to a positive function on S^{aff} which is constant on W-conjugacy classes of simple reflections, and conversely, every such function gives rise to a label function.

We choose a base $\mathbf{q} > 1$ and define $f_s \in \mathbb{R}$ such that $q(s) = \mathbf{q}^{f_s}$ for all $s \in S^{\mathrm{aff}}$.

Given these data, the affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is described as follows. It is the unique complex unital algebra with basis N_w ($w \in W$) over $\mathbb{C}$ subject to the following relations (here $q(s)^{1/2}$ denotes the positive square root of $q(s)$):

(i) $N_{uv} = N_u N_v$ for all $u, v \in W$ such that $l(uv) = l(u) + l(v)$.
(ii) $(N_s + q(s)^{-1/2})(N_s - q(s)^{1/2}) = 0$ for all $s \in S^{\mathrm{aff}}$.

2.0.1. *Root labels for the non-reduced root system.* The label function q on W can also be defined in terms of root labels. We associate a possibly non-reduced root system R_{nr} with $\mathcal{R}$ by

$$R_{\mathrm{nr}} := R_0 \cup \{2\alpha \mid \alpha^\vee \in R_0^\vee \cap 2Y\}. \tag{2.2}$$

Observe that $a + 2 \in Wa$ for all $a \in R$, but that $a + 1 \in Wa$ iff $a = \alpha^\vee + n$ with $2\alpha \notin R_{\mathrm{nr}}$.

Let $R \ni a \to q_a$ be the unique W-invariant function on R such that $q_{a+1} := q(s_a)$ for all $a \in F^{\mathrm{aff}}$. Now for $\alpha = 2\beta \in R_{\mathrm{nr}} \backslash R_0$ we define

$$q_{\alpha^\vee} := \frac{q_{\beta^\vee + 1}}{q_{\beta^\vee}}. \tag{2.3}$$

It is easy to see that in this way the set of positive W_0-invariant functions $\alpha^\vee \to q_{\alpha^\vee}$ on $R_{\mathrm{nr}}^\vee$ corresponds bijectively to the set of label functions q on W. With these conventions we have

for all $w \in W_0$

$$q(w) = \prod_{\alpha \in R_{\mathrm{nr},+} \cap w^{-1} R_{\mathrm{nr},-}} q_{\alpha^\vee}. \tag{2.4}$$

We denote by $R_1 \subset X$ the following reduced root subsystem of R_{nr}:

$$R_1 := \{\alpha \in R_{\mathrm{nr}} \mid 2\alpha \notin R_{\mathrm{nr}}\}. \tag{2.5}$$

Let $F_1 \subset R_1$ be the bases of simple roots corresponding to F_0. The root system R_1 differs from R_0 only if the root datum of $\mathcal{H}$ contains direct summands of "type $C_n^{(1)}$", the irreducible root datum with R_0 of type B_n and X the root lattice of B_n. This is the only irreducible root datum for which the affine Hecke algebra admits 3 independent parameters. For this root datum, R_1 is of type C_n. When applying Lusztig's first reduction theorem (cf. [26, Theorem 2.6]) one needs to consider the affine Weyl group $R_1^{(1)}$ rather than $R_0^{(1)}$, and this is the reason for introducing R_1 (it plays a role in the explicit computations in Chapter 6).

2.0.2. *Restriction to parabolic subsystems.* We define $\mathfrak{a} = Y \otimes_{\mathbb{Z}} \mathbb{R}$. Let P be a subset of F_0. We have a canonical decomposition $\mathfrak{a} = \mathfrak{a}^P \oplus \mathfrak{a}_P$, where $\mathfrak{a}^P := P^\perp$ and $\mathfrak{a}_P := \mathbb{R}P^\vee$. Dually we have the decomposition $\mathfrak{a}^* = \mathfrak{a}^{P,*} \oplus \mathfrak{a}_P^*$ where $\mathfrak{a}_P^* = \mathbb{R}P$ and $\mathfrak{a}^{P,*} = (P^\vee)^\perp$ (in the case $P = F_0$ we will denote this decomposition by $\mathfrak{a}^* = \mathfrak{a}^{0,*} \oplus \mathfrak{a}_0^*$). Let $R_P \subset R_0$ be the "parabolic subsystem of roots" $R_P = R_0 \cap \mathfrak{a}_P^*$.

Consider the root datum $\mathcal{R}^P := (X, Y, R_P, R_P^\vee, P)$. Let $X_P \xrightarrow{\sim} X/(X \cap \mathfrak{a}^{P,*})$ be the projection of the lattice X on $\mathfrak{a}_P^*$ along $\mathfrak{a}^{P,*}$ (this lattice contains the lattice $X \cap \mathfrak{a}_P^*$ as a sublattice of finite index). Observe that the dual lattice Y_P of X_P equals $Y_P = Y \cap \mathfrak{a}_P$. We also introduce the semisimple root datum $\mathcal{R}_P := (X_P, Y_P, R_P, R_P^\vee, P)$. The non-reduced root systems associated to the root data $\mathcal{R}^P$ and $\mathcal{R}_P$ are both equal to $R_{P,\mathrm{nr}} := \mathbb{Q}R_P \cap R_{\mathrm{nr}}$. We define a label function q_P on the affine Weyl group associated to $\mathcal{R}_P$ by requiring that the corresponding root label function on $R_{P,\mathrm{nr}}$ is obtained by restricting the root label function on R_{nr} to $R_{P,\mathrm{nr}}$. We define a label function q^P on the affine Weyl group associated to $\mathcal{R}_P$ in the same fashion.

2.0.3. *Bernstein presentation.* There is a second presentation of the algebra $\mathcal{H}$, due to Joseph Bernstein (unpublished). Since the length function is additive on the dominant cone X^+, the map $X^+ \ni x \mapsto N_x$ is a homomorphism of the commutative monoid X^+ with values in $\mathcal{H}^\times$, the group of invertible elements of $\mathcal{H}$. Thus there exists a unique extension to a homomorphism $X \ni x \mapsto \theta_x \in \mathcal{H}^\times$ of the lattice X with values in $\mathcal{H}^\times$.

Let $\mathcal{A} \subset \mathcal{H}$ be the abelian subalgebra of $\mathcal{H}$ generated by θ_x, $x \in X$. Let $\mathcal{H}_0 = \mathcal{H}(W_0, q_0) \subset \mathcal{H}$ be the finite type Hecke algebra associated with W_0 and the restriction q_0 of q to W_0. Then $\mathcal{H}_0 \subset \mathcal{H}$ is a subalgebra of $\mathcal{H}$. The Bernstein presentation asserts that the multiplication maps $\mathcal{H}_0 \otimes \mathcal{A} \to \mathcal{H}$ and $\mathcal{A} \otimes \mathcal{H}_0 \to \mathcal{H}$ are linear isomorphisms. The algebra structure of $\mathcal{H}$ is then completely determined by the following cross relations (for all $x \in X$ and $s = s_\alpha$ with $\alpha \in F_0$):

$$\begin{aligned} &\theta_x N_s - N_s \theta_{s(x)} = \\ &\begin{cases} (q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee}^{-1/2}) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}} & \text{if } 2\alpha \notin R_{\mathrm{nr}}. \\ ((q_{\alpha^\vee/2}^{1/2} q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee/2}^{-1/2} q_{\alpha^\vee}^{-1/2}) + (q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee}^{-1/2}) \theta_{-\alpha}) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}} & \text{if } 2\alpha \in R_{\mathrm{nr}}. \end{cases} \end{aligned} \tag{2.6}$$

2.0.4. *The center $\mathcal{Z}$ of $\mathcal{H}$.* An immediate consequence of the Bernstein presentation of $\mathcal{H}$ is the description of the center of $\mathcal{H}$:

Theorem 2.1. *The center of $\mathcal{H}$ is equal to $\mathcal{A}^{W_0}$. In particular, $\mathcal{H}$ is finitely generated over its center.*

As an immediate consequence we see that irreducible representations of $\mathcal{H}$ are finite dimensional by an application of (Dixmier's version of) Schur's lemma.

2.1. **Intertwining elements**

Let $s = s_\alpha \in S_0$ with $\alpha \in F_1$. Define $\iota_s \in \mathcal{H}$ by:

$$\begin{aligned}\iota_s &= (1-\theta_{-\alpha})N_s + ((q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1/2} - q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}^{1/2}) + (q_{2\alpha^\vee}^{-1/2} - q_{2\alpha^\vee}^{1/2})\theta_{-\alpha/2})\\ &= N_s(1-\theta_\alpha) + ((q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1/2} - q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}^{1/2})\theta_\alpha + (q_{2\alpha^\vee}^{-1/2} - q_{2\alpha^\vee}^{1/2})\theta_{\alpha/2})\end{aligned}$$

(where, if $\alpha/2 \notin X$, we put $q_{2\alpha^\vee} = 1$). We recall from [24, Theorem 2.8] that these elements of $\mathcal{H}$ satisfy the braid relations, and they satisfy (for all $x \in X$):

$$\iota_s\theta_x = \theta_{s(x)}\iota_s \tag{2.7}$$

Let $\mathcal{Q}$ denote the quotient field of the center $\mathcal{Z}$ of $\mathcal{H}$, and let ${}_{\mathcal{Q}}\mathcal{H}$ denote the $\mathcal{Q}$-algebra ${}_{\mathcal{Q}}\mathcal{H} = \mathcal{Q} \otimes_{\mathcal{Z}} \mathcal{H}$. Inside ${}_{\mathcal{Q}}\mathcal{H}$ we normalize the elements ι_s as follows. We first introduce

$$n_\alpha := q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}^{1/2}(1+q_{\alpha^\vee}^{-1/2}\theta_{-\alpha/2})(1-q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}\theta_{-\alpha/2}) \in \mathcal{A}. \tag{2.8}$$

Then the normalized intertwiners $\iota_s^0 \in {}_{\mathcal{Q}}\mathcal{H}$ ($s \in S_0$) are defined by (with $s = s_\alpha$, $\alpha \in R_1$):

$$\iota_s^0 := n_\alpha^{-1}\iota_s \in {}_{\mathcal{Q}}\mathcal{H}. \tag{2.9}$$

It is known that $(\iota_s^0)^2 = 1$, and in particular that $\iota_s^0 \in {}_{\mathcal{Q}}\mathcal{H}^\times$, the group of invertible elements of ${}_{\mathcal{Q}}\mathcal{H}$. We have:

Lemma 2.2. *([25, Lemma 4.1]) The map $S_0 \ni s \mapsto \iota_s^0 \in {}_{\mathcal{Q}}\mathcal{H}^\times$ extends (uniquely) to a homomorphism $W_0 \ni w \mapsto \iota_w^0 \in {}_{\mathcal{Q}}\mathcal{H}^\times$. Moreover, for all $f \in {}_{\mathcal{Q}}\mathcal{A}$ we have that $\iota_w^0 f \iota_{w^{-1}}^0 = f^w$.*

3. **The Fourier transform for affine Hecke algebras**

Recall the canonical decomposition $\mathfrak{a}^* = \mathfrak{a}^{0,*} \oplus \mathfrak{a}_0^*$. Then $X \cap \mathfrak{a}^{0,*}$ consist of translations of length 0 in the affine Weyl group. Choose a norm $\|\cdot\|$ on $\mathfrak{a}^{0,*}$. Let us denote by x^0 the projection of $x \in X$ onto $\mathfrak{a}^{0,*}$ along $\mathfrak{a}_0^*$. Then we define a norm $\mathcal{N}$ on W by

$$\mathcal{N}(w) := l(w) + \|w(0)^0\|. \tag{3.1}$$

Definition 3.1. *The Schwartz completion $\mathcal{S}$ of $\mathcal{H}$ is the vector space of the formal complex linear combinations $\sum_{w\in W} c_w N_w$ for which the function $W \ni w \to c_w$ is rapidly decreasing with respect to the norm $\mathcal{N}$ defined above on W, equipped with the usual Fréchet topology on the space of rapidly decreasing functions on W.*

Recall the following result:

Theorem 3.2. *([25, Theorem 6.5]) The algebra structure on the dense subspace $\mathcal{H} \subset \mathcal{S}$ extends uniquely to a Fréchet algebra structure on $\mathcal{S}$.*

Let us now review the notions involved in the definition of two of the main ingredients involved in the description of the structure of the Fréchet algebra $\mathcal{S}$, the groupoid $\mathcal{W}_{\Xi_u}$ of standard tempered induction data, and the "induction intertwining functor π" defined on this groupoid. Both these structures arise from the L_2-theory of the Hecke algebra.

3.1. **Tempered representations**

An affine Hecke algebra with a positive label function comes equipped with the structure of a $*$-algebra, where $*$ denotes the unique anti-linear anti-involution defined by anti-linear extension of the map $N_w^* := N_{w^{-1}}$. Moreover, the linear functional τ defined by $\tau(N_w) = \delta_{w,e}$ is a positive trace with respect to $*$. The star operation $*$ and trace τ together define a unique Hilbert algebra structure on $\mathcal{H}$ (see [25]) which is the origin for the harmonic analysis on $\mathcal{H}$.

We define a positive definite Hermitian inner product on $\mathcal{H}$ by $(x, y) := \tau(x^*y)$, and denote by $L_2(\mathcal{H})$ the Hilbert space completion of $\mathcal{H}$. It is the separable Hilbert space in which the basis elements N_w $(w \in W)$ form a Hilbert basis. We have

$$\mathcal{H} \subset \mathcal{S} \subset L_2(\mathcal{H}), \tag{3.2}$$

and the second inclusion is easily seen to be continuous.

A representation π of $\mathcal{H}$ which is of finite length is called *tempered* if π extends continuously to $\mathcal{S}$. It is in fact sufficient that the character of π (recall that all representations of finite length of $\mathcal{H}$ are finite dimensional) extends continuously to $\mathcal{S}$ (cf. [25, Lemma 2.20]). An *irreducible* representation π of $\mathcal{H}$ is called a *discrete series* representation if π extends continuously to $L_2(\mathcal{H})$. An equivalent way of saying this is that the character χ_π of π extends to a continuous functional on $L_2(\mathcal{H})$ (cf. [25, Lemma 2.22]). Thus a discrete series module is in particular a tempered module.

Our main interest in this paper will be the description of the structure of the tempered dual $\hat{\mathcal{S}}$ of $\mathcal{H}$, the set of equivalence classes of irreducible tempered representations. It is known (cf. [9, Theorem 3.11, Theorem 3.19, Theorem 4.3] and [25, Theorem 2.25]) that this set of irreducible representations extends to the C^*-algebra completion $C_r^*(\mathcal{H})$ of $\mathcal{H}$, and that one obtains in this way precisely the irreducible spectrum of $C_r^*(\mathcal{H})$. We equip $\hat{\mathcal{S}}$ with the topology of the spectrum of $C_r^*(\mathcal{H})$ via this identification.

3.2. **The groupoid of standard induction data**

3.2.1. *Induced representations.* The affine Hecke algebra with root datum $\mathcal{R}^P$ and label function q^P is naturally embedded as a subalgebra of $\mathcal{H}$, as is apparent from Bernstein's presentation. The affine Hecke algebra with root datum $\mathcal{R}_P$ and label function q_P is isomorphic to the quotient of $\mathcal{H}^P$ by the central subalgebra $\mathcal{A}^P \subset \mathcal{H}^P$ generated by the θ_x with $x \in X \cap \mathfrak{a}^{P,*}$. In particular for $\mathcal{H}_P$ the rôle of the complex algebraic torus T of (quasi) characters of X is now played by the algebraic subtorus $T_P \subset T$ with character lattice X_P. The central characters of the irreducible modules over $\mathcal{H}_P$ are W_P-orbits in T_P.

Let $T^P \subset T$ denote the complex algebraic subtorus with character lattice $X^P = X/(X \cap \mathfrak{a}_P^*)$. This is the identity component of the group of fixed points for the action of W_P on T. The group T^P acts naturally on $\mathcal{H}^P$ by automorphisms. We send $t \in T^P$ to the automorphism ψ_t of $\mathcal{H}^P$ which acts on the Bernstein basis by $\theta_x N_w \to t(x)\theta_x N_w$. Given a discrete series representation δ of $\mathcal{H}_P$ we denote by δ_t the twist $\delta_t = \delta \circ p \circ \psi_t$ by $t \in T^P$ of the lift of δ to

$\mathcal{H}^P$ via the natural quotient map $p : \mathcal{H}^P \to \mathcal{H}_P$. Define the finite abelian group

$$K_P := T^P \cap T_P \approx \mathrm{Hom}(X_P/(X \cap \mathfrak{a}_P^*), \mathbb{C}^\times) \tag{3.3}$$

For later use we observe that if $k \in K_P$ then the twist by k descends to an automorphism of $\mathcal{H}_P$, and we have $\delta_{tk} = (\delta_k)_t$, where $\delta_k (= \delta^{k^{-1}})$ is the twist of δ by the automorphism of $\mathcal{H}_P$ coming from k, which is again a discrete series representation of $\mathcal{H}_P$.

Choose a complete set of representatives $\Delta_P = \Delta_{\mathcal{R}_P, q_P}$ for the set of isomorphism classes of discrete series representations of the Hecke algebra $\mathcal{H}_P$. This set is finite [25, Lemma 3.31]. We put Δ for the finite disjoint union $\Delta := \coprod_{P\in\mathcal{P}} (P, \Delta_P)$, a finite set with a natural fibration $\Delta \to \mathcal{P}$ (with $\mathcal{P}$ the power set of F_0).

We will use some terminology from the theory of groupoids. Recall that a groupoid $\mathcal{G}$ is a "group with several objects" or more formally, a small category in which all the morphisms are invertible. In particular a group G is a groupoid with one object. On the other extreme end any set X can be viewed as a groupoid with only identities, the "identity groupoid" of X.

A standard induction datum ξ for $\mathcal{H}$ is a triple (P, δ, t) with $P \in \mathcal{P}$, $\delta \in \Delta_P := \Delta_{\mathcal{R}_P, q_P}$, and $t \in T^P$. Recall that δ is a representative of an equivalence class of discrete series representations. Let us denote the underlying vector space by V_δ. The set Ξ of all such triples is a finite (by [25, Lemma 3.31]) disjoint union of the subsets $\Xi_{(P,\delta)}$, each of which is a copy of the complex algebraic torus T^P. We view Ξ as the set of arrows of a groupoid whose set of objects is Δ, with $\mathrm{Hom}((P, \delta), (Q, \tau)) = \Xi_{(P,\delta)}$ if $(P, \delta) = (Q, \tau)$ and $= \emptyset$ else. We identify $\Xi_{(P,\delta)}$ with the complex algebraic torus T^P by $T^P \ni t \to \xi = (P, \delta, t) \in \Xi_{(P,\delta)}$. This equips Ξ in particular with the structure of a complex algebraic variety. We denote by $\Xi_u \subset \Xi$ the compact real form of Ξ (i.e. we restrict t to the compact real form $T_u^P \subset T^P$). Given an induction datum $\xi = (P, \delta, t)$ we can define an *induced representation* $\pi(\xi)$ of $\mathcal{H}$ by inducing δ_t from $\mathcal{H}^P$ to $\mathcal{H}$ (see [25, Paragraph 4.5.1]; [9, Subsection 3.5]). The representation is realized in the vector space

$$V_\xi = \mathcal{H} \otimes_{\mathcal{H}^P} V_\delta =: i(V_\delta) \tag{3.4}$$

which is *independent of* $t \in T^P$ (the "compact realization"). The matrix coefficients of $\pi(\xi)$ are regular functions on Ξ. The representations $(\pi(\xi), V_\xi)$ are called *generalized principal series* representations.

Proposition 3.3. *(*[25, Proposition 4.19, Proposition 4.20]*) The generalized principal series $\pi(\xi)$ is tempered if $\xi \in \Xi_u$, and it is unitary for $\xi \in \Xi_u$ with respect to a standard inner product on $i(V_\delta)$ which is independent of ξ.*

3.2.2. *The groupoid of standard induction data.* We now describe the morphisms of standard induction data. Recall the Weyl groupoid $\mathfrak{W}$ which has the collection of standard parabolic subsets $P \subset F_0$ as set of objects, with arrows $\mathfrak{W}_{P,Q} := \{w \in \mathfrak{W} | w(P) = Q\}$ between two standard parabolic subsets P, Q (see the Appendix 7 for some important notions related to $\mathfrak{W}$). If $w \in \mathfrak{W}_{P,Q}$ then there exists a corresponding isomorphism of root data $\mathcal{R}_P \to \mathcal{R}_Q$ compatible with the root labels q_P and q_Q, thus defining an isomorphism $\psi_w : \mathcal{H}_P \to \mathcal{H}_Q$. On the other hand, we have already seen above that with $k \in K_Q$ there is associated a twist ψ_k of $\mathcal{H}_Q$. We define a groupoid $\mathcal{W}$ whose set of objects is $\mathcal{P}$ and $\mathcal{W}_{P,Q} := K_Q \times \mathfrak{W}_{P,Q}$ with the obvious composition rule $(k \times u) \circ (l \times v) = k(u(l)) \times uv$. We also introduce the "normal subgroupoid" $\mathcal{K} \subset \mathcal{W}$ whose objects are $\mathcal{P}$, with $\mathcal{K}_{P,Q} = \emptyset$ if $P \neq Q$ and $\mathcal{K}_{P,P} = K_P$. Hence the Weyl groupoid $\mathfrak{W}$ is equal to the quotient $\mathfrak{W} = \mathcal{W}/\mathcal{K}$.

For each $g = k \times u \in \mathcal{W}$ we define an isomorphism $\psi_g : \mathcal{H}_P \to \mathcal{H}_Q$ by $\psi_g = \psi_{k\times u} := \psi_k \circ \psi_u$. There is a natural action of $\mathcal{W}$ on the space Ξ by (with $g = k \times u \in K_Q \times \mathfrak{W}_{P,Q}$):

$$g(P, \delta, t) := (u(P), \delta^g, g(t)), \tag{3.5}$$

where $\delta^g \in \Delta_Q$ is the unique discrete series representation such that $\delta^g \simeq \delta \circ \psi_g^{-1}$.

Definition 3.4. *We define the groupoid $\mathcal{W}_\Xi$ of standard induction data by $\mathcal{W}_\Xi := \mathcal{W} \times_{\mathcal{P}} \Xi$. Its set of objects is Ξ, and the morphisms in $\mathcal{W}_\Xi$ from $\xi \to \eta$ are the $g \in \mathcal{W}$ such that $g(\xi) = \eta$. The full subgroupoid $\mathcal{W}_{\Xi_u}$ is obtained by restricting the set of objects to $\Xi_u \subset \Xi$.*

Notice that the groupoid $\mathcal{W}_\Xi$ is canonically determined by $\mathcal{H}$. In particular $\mathcal{W}_\Xi$ is independent of the chosen representatives of the isomorphism classes of discrete series representations Δ.

Given a morphism $g = k \times u \in \mathcal{W}_{P,Q}$ in the groupoid and a discrete series representation $\delta \in \Delta_P$, we now *choose* an isomorphism

$$\tilde{\delta}_g : V_\delta \to V_{\delta^g} \tag{3.6}$$

intertwining the irreducible representations $\delta \circ \psi_g^{-1}$ and δ^g. Given $\xi = (P, \delta, t) \in \Xi_(P, \delta)$ we will define a normalized intertwining operator

$$\pi(g, \xi) : (\pi(\xi), i(V_\delta)) \to (\pi(g\xi), i(V_{\delta^g})) \tag{3.7}$$

under certain regularity conditions on ξ (see the discussion below; for further detail we refer to[25, Section 4.4] and to [9, equation (3.8)]). The definition is complicated since it involves the intertwining elements $\iota^0_{u^{-1}} \in {}_Q\mathcal{H}$, which act in a representation $\pi(\xi)$ only if ξ is such that the poles of the intertwining elements are avoided. In [25, Section 4.4] the normalized intertwining operators $\pi(g, \xi)$ are first defined algebraically in the Zariski open set of the so-called R_P-generic elements $\xi \in \Xi_{(P,\delta)}$, and afterwards extended to a larger open set in the analytic topology containing $\Xi_{(P,\delta),u}$, using the unitarity of the $\pi(g, \xi)$. The element $\xi = (P, \delta, t) \in \Xi_{(P,\delta),u}$ is called R_P-generic if the orbit $W_P r_\delta t \subset T$ consists of R_P-generic elements in the sense of [25, Definition 4.12], where $W_P r_\delta \subset T_P$ denotes the central character of δ. The set of such R_P-generic ξ is Zariski-open in $\Xi_{(P,\delta)}$, and for ξ in this set we can define the normalized intertwining operator on $\pi(\xi)$ by the formula

$$\pi(g, \xi)(N_w \otimes v) = \pi(u(\xi), N_w)\pi(u(\xi), \iota^0_{u^{-1}})(1 \otimes \tilde{\delta}_g(v)), \tag{3.8}$$

(see [25, Section 4.4], or [9, Section 3.5]). However, in a suitable open neighborhood of $\Xi_{(P,\delta),u}$ the apparent poles of the normalized intertwining operators turn out to be removable (see [25, subsection 4.4]). Hence we can uniquely extend the $\pi(g, \xi)$ to a smooth family of operators depending on ξ in a suitable open neighborhood of $\Xi_{(P,\delta),u}$. The normalized intertwining operators on Ξ_u we use in the present paper are the restrictions to Ξ_u of these regular rational functions of ξ defined in a neighbourhood of $\Xi_u \subset \Xi$. They are in fact unitary for $\xi \in \Xi_u$ with respect to the standard inner products on $i(V_\xi)$ and $i(V_{g(\xi)})$ (cf. [25, Proposition 4.19]). In this way we have obtained the following result (see [25, Theorem 3.38] and [9, Theorem 3.14]):

Theorem 3.5. *The assignment $\Xi_{(P,\delta),u} \ni \xi \to \pi(\xi)$ and $\mathcal{W}_{P,Q} \ni g \to \pi(g, \xi)$ extends to a functor π (the "induction intertwining" functor) from $\mathcal{W}_{\Xi_u}$ to $\mathbb{P}\operatorname{Rep}(\mathcal{H})_{temp,unit}$, the category of tempered, unitary modules of $\mathcal{H}$ in which the morphisms are unitary $\mathcal{H}$-intertwiners modulo scalars. The functor π is rational and regular in $\xi \in \Xi_u$.*

Theorem 3.6. *(*[9, Theorem 3.19 and Corollary 5.6]*) Any irreducible tempered representation V is isomorphic to a summand of a generalized principal series representation for a unitary standard induction datum $\xi \in \Xi_u$ whose isomorphism class is uniquely determined by V.*

This theorem tells us that in order to classify the irreducible tempered representations, it is enough to classify the discrete series representations and to understand how the generalized principal series representations with unitary induction parameter decompose in irreducible subrepresentations. The theory of the analytic R-group below is designed to resolve this last problem of the decomposition of $\pi(\xi)$ for unitary ξ.

3.2.3. *The 2-cocycle $\gamma_{\mathcal{W},\Delta}$.* It is conventional to denote by $\mathcal{G}^0$ the set of objects of a groupoid $\mathcal{G}$, and by $\mathcal{G}^1$ the set of morphisms or arrows of $\mathcal{G}$. Each arrow $g \in \mathcal{G}^1$ has a source object $s(g)$ and a target object $t(g)$, and this defines two maps $s, t : \mathcal{G}^1 \to \mathcal{G}^0$. The set of composable pairs of arrows is $\mathcal{G}^2 := \{(g_1, g_2) \mid g_i \in \mathcal{G}^1, s(g_2) = t(g_1)\}$. This set is thus is a fibered product

$$\mathcal{G}^2 = \mathcal{G}^1{}_s \times_t \mathcal{G}^1. \tag{3.9}$$

The twisting isomorphisms $\psi_g : \mathcal{H}_P \to \mathcal{H}_Q$ with $g \in \mathcal{W}_{P,Q}$ define a homomorphism from the groupoid $\mathcal{W}$ to the groupoid $\mathrm{Iso}_{\mathcal{P}}$ whose set of objects is $\mathcal{P}$ and whose morphisms $\mathrm{Iso}_{\mathcal{P}}(P, Q)$ consist of algebra isomorphisms from $\mathcal{H}_P$ to $\mathcal{H}_Q$ which map $\mathcal{A}_{P,+}$ to $\mathcal{A}_{Q,+}$, where $\mathcal{A}_{P,+}$ is the subalgebra of $\mathcal{H}_P$ spanned by the elements N_x with $x \in X_{P,+}$ and similarly for $\mathcal{A}_{Q,+}$. This induces an action of $\mathcal{W}$ on the set Δ. The choice of intertwining isomorphisms made in (3.6) determines a $\mathrm{U}(1)$-valued 2-cocycle $\gamma_{\mathcal{W},\Delta}$ on the finite groupoid $\mathcal{W}_\Delta = \mathcal{W} \times_{\mathcal{P}} \Delta$ (which has the finite set Δ as its set of objects) as follows. Let g', g be composable arrows in $\mathcal{W}$, let $\delta \in \Delta_P$ where P is the source of g, and let $\delta' = \delta^g$. With the above notations, we have $((g', \delta'), (g, \delta)) \in \mathcal{W}_\Delta^2 := \mathcal{W}_{\Delta s} \times_t \mathcal{W}_\Delta$ (where s, t denote the source and target map of the groupoid $\mathcal{W}_\Delta$). We define a function $\gamma_{\mathcal{W},\Delta}$ on $\mathcal{W}_\Delta^2$ by:

$$\tilde{\delta}'_{g'} \circ \tilde{\delta}_g = \gamma_{\mathcal{W},\Delta}((g', \delta'), (g, \delta))\tilde{\delta}_{g'g}. \tag{3.10}$$

Proposition 3.7. *The function $\gamma_{\mathcal{W},\Delta}$ defines a 2-cocycle on $\mathcal{W}_\Delta^2$ with values in $\mathrm{U}(1)$, whose class $[\gamma_{\mathcal{W},\Delta}] \in H^2(\mathcal{W}_\Delta, \mathrm{U}(1))$ is independent of the choices of the $\tilde{\delta}_g$. We can choose the $\tilde{\delta}_g$ such that $\gamma_{\mathcal{W},\Delta}$ has values in the group $\mu(D_\Delta)$ of complex D_Δ-th roots of unity, with $D_\Delta := \mathrm{lcm}_{\delta\in\Delta}\{\dim(V_\delta)\}$.*

Proof. Let $((f, \delta), (g, \epsilon), (h, \zeta)) \in \mathcal{W}_\Delta^3 := \mathcal{W}_{\Delta s} \times_t \mathcal{W}_{\Delta s} \times_t \mathcal{W}_\Delta$. By associativity

$$(\tilde{\delta}_f \circ \tilde{\epsilon}_g) \circ \tilde{\zeta}_h = \tilde{\delta}_f \circ (\tilde{\epsilon}_g \circ \tilde{\zeta}_h) \tag{3.11}$$

one checks the 2-cocycle relation of $\gamma_{\mathcal{W},\Delta}$. It is clear that changing the choices of the isomorphisms $\tilde{\delta}_g$ in equation (3.6) changes $\gamma_{\mathcal{W},\Delta}$ only by a coboundary. Let us now prove the last assertion. First suppose that $D_\Delta = 1$. Choose a basis vector in V_δ for each pair (P, δ) with $\delta \in \Delta_P$. Then $\tilde{\delta}_g$ is a complex scalar, and relation (3.10) expresses $\gamma_{\mathcal{W},\Delta}$ as the coboundary of the $\mathbb{C}^\times$-valued function $g \to \tilde{\delta}_g$ on $\mathcal{W}_\Delta^1$, proving the assertion in this special case. In the general case, taking determinants (and suitable powers) in (3.10) similarly shows that $\gamma_{\mathcal{W},\Delta}^{D_\Delta}$ is the coboundary of a $\mathbb{C}^\times$-valued function ϵ on $\mathcal{W}_\Delta^1$, i.e. $\gamma_{\mathcal{W},\Delta}(g, g')^{D_\Delta} = \epsilon(g)\epsilon(g')\epsilon(g \circ g')^{-1}$ for all $(g, g') \in \mathcal{W}_\Delta^2$. Now choose $\zeta(g)$ for $g \in \mathcal{W}_\Delta$ such that $\epsilon(g) = \zeta^{D_\Delta}(g)$, and replace $\tilde{\delta}_g$ by $\tilde{\delta}'_g = \zeta^{-1}\tilde{\delta}_g$. The 2-cocycle $\gamma'_{\mathcal{W},\Delta}$ defined by (3.10) after replacing $\tilde{\delta}_g$ by $\tilde{\delta}'_g$ takes values in $\mu(D_\Delta)$. ∎

The projective representation π of the groupoid $\mathcal{W}_{\Xi_u}$ is related to $\gamma_{\mathcal{W},\Delta}$ by the following formula, which follows immediately from the definition of the cocycle $\gamma_{\mathcal{W},\Delta}$, the definition of the normalized intertwining operators, and from the fact that the normalized intertwining elements satisfy the Weyl group relations (cf. [25, Lemma 4.1]): Let $\xi = (P, \delta, t) \in \Xi_{(P,\delta),u}$, then

$$\pi(h, g\xi) \circ \pi(g, \xi) = \gamma_{\mathcal{W},\Delta}((h, \delta^g), (g, \delta))\pi(hg, \xi) \tag{3.12}$$

Definition 3.8. *Formula (3.12) defines a torsion* 2*-cohomology class* $[\gamma_{\mathcal{W},\Xi}] \in H^2(\mathcal{W}_{\Xi_u}, \mathrm{U}(1))$, *the pull back of* $[\gamma_{\mathcal{W},\Delta}]$ *via the natural homomorphism of groupoids* $\mathcal{W}_{\Xi_u} \to \mathcal{W}_\Delta$.

One should think of $[\gamma_{\mathcal{W},\Xi}]$ as the characteristic class of the projective bundle over the groupoid $\mathcal{W}_\Xi$ which is defined by the induction intertwining functor π.

3.2.4. *Inertial orbits of discrete series modulo the center.* It is sometimes convenient to work with a slightly modified version of the groupoid of standard induction data.

We say that an irreducible representation σ of $\mathcal{H}^P$ is discrete series modulo the center if σ is equivalent to a representation of the form δ_t with $t \in T_u^P$ and δ an irreducible discrete series representation of $\mathcal{H}_P$ (see paragraph 3.2.1). Let $P \subset F_0$ and let σ be a discrete series representation modulo the center of $\mathcal{H}^P$. By definition the inertial orbit $\mathcal{O}_{(P,\sigma)}$ of σ is the set of the equivalence classes $(P, \sigma \circ \psi_t)$ of the $\mathcal{H}^P$-representations $\sigma \circ \psi_t$ where t varies in T^P. This gives a natural T^P action on $\mathcal{O}_{(P,\delta)}$.

It is obvious that the isotropy of a datum $(P, \sigma) \in \mathcal{O}_{(P,\delta)}$ is always a finite subgroup of T^P. It is also clear (see also the discussion in paragraph 3.2.1) that each orbit $\mathcal{O}_{(P,\sigma)}$ contains a unique K_P-orbit of discrete series modulo the center which descend to $\mathcal{H}_P$. If σ_P descends to $\mathcal{H}_P$ then there exists a discrete series representation δ of $\mathcal{H}_P$ such that $[\delta_1] = \sigma_P$. Therefore there exists, for each orbit $\mathcal{O}_{(P,\sigma)}$, a component $\Xi_{(P,\delta)}$ of Ξ and a finite covering map

$$\begin{aligned} \Xi_{(P,\delta)} &\to \mathcal{O}_{(P,\sigma)} \\ (P, \delta, t) &\to (P, [\delta_t]) \end{aligned} \tag{3.13}$$

In this case we will also use the notation $\mathcal{O}_{(P,\delta)}$ to denote $\mathcal{O}_{(P,\sigma)}$. It is easy to see that for given $\xi = (P, \delta, t)$ and $\xi' = (P, \delta', s)$ we have $[\delta'_s] = [\delta_t]$ (isomorphic as representations of $\mathcal{H}^P$) if and only if $K_P\xi = K_P\xi'$. In other words, we have

$$\mathcal{O} = \mathcal{K}\backslash\Xi = |\mathcal{K}_\Xi|, \tag{3.14}$$

(where $|\mathcal{K}_\Xi|$ denotes the orbit space of isomorphism classes of objects of the groupoid $\mathcal{K}_\Xi = \mathcal{K} \times_{\mathcal{P}} \Xi$), and the covering (3.13) is given by taking the quotient of $\Xi_{(P,\delta)}$ by the isotropy subgroup $K_\delta \in K_P$ of $[\delta]$. Since the action of $\mathcal{K}$ on Ξ is free, the orbit map extends to a homomorphism of groupoids (viewing $\mathcal{O}$ as the "unit" groupoid with only identity morphisms)

$$\mathcal{K}_\Xi \to \mathcal{O} \tag{3.15}$$

which is a Morita equivalence (in the sense of [20]). The space $\mathcal{O}$ is a disjoint union of finitely many orbits of the form $\mathcal{O}_{(P,\delta)}$ (parameterized by the $\mathcal{K}$-orbits on Δ), and each orbit $\mathcal{O}_{(P,\delta)}$ has the natural structure of a T^P/K_δ-torsor (corresponding to the multiplication action of T^P on $\Xi_{(P,\delta)}$ by identifying $\Xi_{(P,\delta)}$ with T^P). This gives $\mathcal{O}$ the structure of a complex algebraic variety and it defines a special compact form $\mathcal{O}_u$ of $\mathcal{O}$.

Clearly $\mathcal{O}$ carries a natural action of the Weyl groupoid $\mathfrak{W} = \mathcal{W}/\mathcal{K}$. We consider the groupoid

$$\mathfrak{W}_{\mathcal{O}} := \mathfrak{W} \times_{\mathcal{P}} \mathcal{O} \tag{3.16}$$

(and its compact form $\mathcal{W}_{\mathcal{O}_u}$). The observations made in this paragraph amount to saying that:

Proposition 3.9. *The groupoids $\mathcal{W}_\Xi$ ($\mathcal{W}_{\Xi_u}$) and $\mathfrak{W}_\mathcal{O}$ (resp. $\mathfrak{W}_{\mathcal{O}_u}$) are Morita equivalent.*

However, it is important to observe at this point that:

Remark 3.10. *Let $|\Delta|$ denote the set of isomorphism classes of the normal subgroupoid $\mathcal{K}_\Delta$ of $\mathcal{W}_\Delta$. The quotient homomorphism $\mathcal{W}_\Delta \to \mathfrak{W}_{|\Delta|}$ (defined by sending $w \times k \to w$ and $\delta \to \mathcal{K}\delta$) is a Morita equivalence if and only if all the isotropy groups K_δ are trivial.*

3.3. **The Fourier isomorphism**

We will formulate the main result of [9] (see loc.cit Section 5) in this section. Denote by the trivial vector bundle over Ξ whose fibre at ξ is equal to $V_\xi = i(V_\delta)$, thus

$$\mathcal{V}_\Xi := \coprod_{(P,\delta)} \Xi_{P,\delta} \times i(V_\delta) \tag{3.17}$$

The algebra of smooth sections of the trivial bundle $\mathrm{End}(\mathcal{V}_\Xi)$ on Ξ_u will be denoted by $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))$. We equip this algebra with its usual Fréchet topology. We define the set of $\mathcal{W}$-equivariant sections in this bundle as follows. Recall that $\pi(g, \xi)$ is smooth and has smooth inverse on Ξ_u. Take $\xi \in \Xi_{P,u}$ and let A be an element of $\mathrm{End}(V_\xi)$. For $g \in \mathcal{W}_\xi$ (where W_ξ denotes the set of elements in $\mathcal{W}$ which act on ξ, hence with source P) we define $g(A) := \pi(g, \xi) \circ A \circ \pi(g, \xi)^{-1} \in \mathrm{End}(V_{g(\xi)})$.

Definition 3.11. *A section of f of $\mathrm{End}(\mathcal{V}_\Xi))$ is called $\mathcal{W}$-equivariant if we have $f(\xi) = g^{-1}(f(g(\xi)))$ for all $\xi \in \Xi$ and $g \in \mathcal{W}_\xi$. We denote the subalgebra of smooth $\mathcal{W}$-equivariant sections by $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))^\mathcal{W}$.*

The Fourier transform $\mathcal{F}$ is canonically defined in terms of the induction intertwining functor π: Given $x \in \mathcal{S}$ we define a section $\mathcal{F}(x)$ of $\mathrm{End}(\mathcal{V}_\Xi)$ by $\mathcal{F}(x)(\xi) := \pi(\xi, x)$. The fact that the target of π is a category whose objects are unitary representations of $\mathcal{H}$ implies that $\mathcal{F}$ is an algebra homomorphism, and the functoriality of π amounts to the fact that $\mathcal{F}(x)$ is a $\mathcal{W}$-equivariant section in the above sense. In [9, Proposition 7.3] it was shown that in fact $\mathcal{F}(\mathcal{S}) \subset C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))^\mathcal{W}$ (this inclusion is not very hard to prove).

We define a wave packet operator at first as the isometry

$$\mathcal{J} : L_2(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi), \mu_{Pl}) \to L^2(\mathcal{H}) \tag{3.18}$$

(where μ_{Pl} is the Plancherel measure, cf. [9, Section 4]) which is the adjoint of the L_2-extension of the Fourier transform. From the expression of the density function of the Plancherel measure it is easy to see that the space

$$\mathcal{C}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)) := cC^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)), \tag{3.19}$$

where c denotes the c-function on Ξ_u (see e.g. [9, Definition (9.7)]), is a subspace of the Hilbert space $L_2(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi), \mu_{Pl})$. Hence $\mathcal{J}$ is well defined on this vector space. We equip $\mathcal{C}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))$ with the Fréchet topology of $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))$ via the linear isomorphism $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)) \to \mathcal{C}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))$ defined by $\sigma \to c\sigma$. Finally we define an averaging projection $p_\mathcal{W}$ onto the space of $\mathcal{W}$-equivariant sections by:

$$p_\mathcal{W}(f)(\xi) := |\mathcal{W}_\xi|^{-1} \sum_{g \in \mathcal{W}_\xi} g^{-1}(f(g(\xi))). \tag{3.20}$$

We can now formulate the main result of [9]:

Theorem 3.12. *The Fourier transform restricts to an isomorphism of Fréchet algebras*

$$\mathcal{F} : \mathcal{S} \to C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))^{\mathcal{W}}. \tag{3.21}$$

The wave packet operator $\mathcal{J}$ restricts to a surjective continuous map

$$\mathcal{J}_\mathcal{C} : \mathcal{C}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)) \to \mathcal{S}. \tag{3.22}$$

We have $\mathcal{J}_\mathcal{C}\mathcal{F} = \mathrm{id}_\mathcal{S}$, and we have $\mathcal{F}\mathcal{J}_\mathcal{C} = p_\mathcal{W}|_{\mathcal{C}(\Xi_u,\mathrm{End}(\mathcal{V}_\Xi))}$. In particular, the map $p_\mathcal{W}$ is a continuous projection of $\mathcal{C}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))$ onto $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))^{\mathcal{W}}$.

The projection $p_\mathcal{W}$ thus cancels singularities of sections over Ξ_u which are no worse than the poles of the c-function on Ξ_u. This property of $p_\mathcal{W}$ is crucially important in the sequel of the paper.

4. The analytic R-group

In this section we will define the notion of the analytic R-group $\mathfrak{R}_\xi$ in our context for a given unitary standard induction datum $\xi \in \Xi_u$. Our treatment follows closely the argument of [30] but is more direct. For a good account of the rôle of the R-group in the work of Harish-Chandra and of Knapp and Stein [14] we refer the reader to [1, Section 2].

The group $\mathfrak{R}_\xi$ is a subgroup of the inertia group $\mathcal{W}_{\xi,\xi}$ which is a complement of a certain normal reflection subgroup $\mathcal{W}^m_{\xi,\xi}$ of $\mathcal{W}_{\xi,\xi}$. The reflection hyperplanes of the reflections in $\mathcal{W}^m_{\xi,\xi}$ are described in terms of the Plancherel density function. The importance of the R-group $\mathfrak{R}_\xi$ is that the induced module $\pi(\xi)$ (which naturally comes with the structure of a $\mathcal{H} - {}^\gamma\mathbb{C}[\mathcal{W}_{\xi,\xi}]^{op}$ bimodule via the induction-intertwining functor) is a Morita equivalence module between the opposite of the γ_ξ twisted group ring of $\mathfrak{R}_\xi$ (for a certain 2-cocycle γ_ξ derived from $\gamma_{\mathcal{W},\Xi}$) on the one hand, and the category of tempered unitary $\mathcal{H}$-modules with central character $\mathcal{W}\xi$ (in the sense of a character of the center of the Schwartz algebras $\mathcal{S}$, see [9, Corollary 5.5]) on the other hand. This implies in particular that the irreducible tempered modules of $\mathcal{H}$ with central character $\mathcal{W}\xi$ are in one-to-one correspondence with the irreducible characters of ${}^\gamma\mathbb{C}[\mathfrak{R}_\xi]$ (see [1, Section 2]).

4.0.1. *Definition of the R-group.* We identify $\Xi_{(P,\delta)}$ with the complex torus T^P, and in doing so, we in particular give meaning to group theoretical operations in $\Xi_{(P,\delta)}$ (such as ξ^{-1}). Below we use notations and concepts associated to the chamber system of the Weyl groupoid and restrictions of roots to facets of the Weyl chamber; we refer the reader to Section 7 for these notations and some basic facts. We also recall the decomposition $\mathfrak{a} = \mathfrak{a}^P \oplus \mathfrak{a}_P$ (see paragraph 2.0.2) for $P \in \mathcal{P}$.

The rational function $\nu(\xi) = (c(\xi)c(\xi^{-1}))^{-1}$ (which is the density function for the Plancherel measure, up to normalizing constants) is known to be regular and positive on Ξ_u (cf. [9, Proposition 9.8]). This applies to the corank 1 factors of the c-function as well, so by the product formula for ν (see [9, Definition 9.7]) it is clear that the zero set of ν in $\Xi_{(P,\delta),u}$ (with $(P,\delta) \in \Delta$) is a finite union of orbits $M_{(P,\alpha),\xi,u}$ (with $(P,\alpha) \in R^P$ and $\xi \in \Xi_{(P,\delta),u}$) of codimension 1 subtori of the form $T_u^{(P,\alpha)} \subset T_u^P$, the unique codimension one subtorus which lies in the kernel of the character $(P,\alpha) \in R^P$.

Definition 4.1. *The orbits of the form $M_{(P,\alpha),\xi,u}$ in the zero set of ν intersected with Ξ_u are called mirrors in Ξ_u. The collection of all mirrors is denoted by $\mathcal{M}$. The set of mirrors in $\Xi_{(P,\delta),u}$ is denoted by $\mathcal{M}_{(P,\delta)}$, so that $\mathcal{M} = \coprod_{(P,\delta)\in\Delta} \mathcal{M}_{(P,\delta)}$ (a disjoint union).*

Proposition 4.2. *The collection $\mathcal{M}$ is $\mathcal{W}$-invariant.*

Proof. This is clear by the $\mathcal{W}$-invariance of ν. □

The next theorem is inspired by well known results of Harish-Chandra (see [10, Section 39], and also [14], [30]):

Theorem 4.3. *Let $M \in \mathcal{M}_{(P,\delta)}$.*

(i) *There exists a unique involution $\mathfrak{s}_M \in \mathcal{W}_{(P,\delta),(P,\delta)}$ (the inertia group in $\mathcal{W}_\Delta$ of the object (P,δ)) such that $\mathfrak{s}_M$ leaves M pointwise fixed.*

(ii) *$\mathfrak{s}_M$ is $\mathcal{W}$-conjugate to an element of the form $\mathfrak{s}' \times k'$ with $\mathfrak{s}' = \mathfrak{s}_{Q'}^{P'} \in \mathfrak{W}_{P',P'}$ an elementary conjugation with $P' \subset Q'$ self-opposed (see paragraph 7.0.4), and $k' \in K_{P'}$ such that $\mathfrak{s}' \times k' = (k')^{-1} \times \mathfrak{s}'$.*

(iii) *The rational function $\xi \to c(\xi)$ on $\Xi_{(P,\delta),u}$ has a pole of order one at M.*

(iv) *For all $\xi \in M$, the intertwining operator $\pi(\mathfrak{s}_M, \xi)$ is a scalar.*

The element $\mathfrak{s}_M$ is called the reflection in M.

Proof. The proof of this result is based on the following aspect of the main Theorem 5.3 of [9]. Let $V = i(V_\delta)$ denote the vector space on which all the induced representations $\pi(\xi)$ (with $\xi \in \Xi_{(P,\delta)}$) are realized in the compact realization. Let

$$f : \Xi_{(P,\delta),u} \to \mathrm{End}(V) \tag{4.1}$$

be a smooth section, and extend this function by 0 on the other components of Ξ_u. Then Theorem 5.3 of [9] implies that the function $p_{\mathcal{W}}(cf)$ on Ξ_u (where c denotes the c-function on Ξ_u) defined by

$$p_{\mathcal{W}}(cf)(\xi) = |\mathcal{W}_\xi|^{-1} \sum_{g\in\mathcal{W}_\xi} \pi(g,\xi)^{-1}(cf(g(\xi)))\pi(g,\xi) \tag{4.2}$$

is again smooth on Ξ_u.

Recall that (by the Maass Selberg relations, see [9, Proposition 9.8]) the function c vanishes on M since ν vanishes on M. Let $\xi \in M$ and let $\mathcal{W}_{\xi,\xi} \subset \mathcal{W}_{(P,\delta),(P,\delta)}$ denote the subgroup of elements which fix ξ. If the identity is the only element of $\mathcal{W}$ which fixes the elements of M pointwise then $\mathcal{W}_{\xi,\xi} = \{e\}$ for generic $\xi \in M$. In that case there would exist a small open neighborhood $U \ni \xi$ such that $wU \cap U = \emptyset$ if $w \in \mathcal{W}_\xi$ but $w \neq e$. Hence if we take f such that its support is contained in U but with $f(\xi) \neq 0$ the expression (4.2) will not be smooth on U, a contradiction.

We conclude that there exists an element $\mathfrak{s} \in \mathcal{W}_{(P,\delta),(P,\delta)}$ which fixes M pointwise and which is not the identity on $\Xi_{(P,\delta)}$. Thus locally in the tangent space $i\mathfrak{a}^P$ of $\Xi_{(P,\delta)}$ at $\xi \in M$, $\mathfrak{s}$ must be given by an element of $\mathfrak{W}_{P,P}$ that fixes the hyperplane in $i\mathfrak{a}^P$ which corresponds to M under the exponential mapping (where we choose ξ as the identity element of $\Xi_{(P,\delta)}$). This uniquely determines $\mathfrak{s}$ on $\Xi_{(P,\delta),u}$ and shows that $\mathfrak{s}$ is an involution. It also follows that $\mathfrak{s}$ is $\mathcal{W}$-conjugate to an involutive elementary conjugation (see paragraph 7.0.4) composed with an element of $\mathcal{K}$ such that the composition is still an involution, proving both (i) and (ii).

Let us now consider (iii). Take a generic element $\xi = (P, \delta, t_0)$ of M such that $\mathcal{W}_{\xi,\xi} = \{e, \mathfrak{s}_M\}$ and let U be a small open neighborhood of ξ which is invariant for $\mathfrak{s}_M$ and has the

property that $wU \cap U \neq \emptyset$ iff $w(\xi) = \xi$. For $t \in T_u^P$ we write $\xi_t = (P, \delta, tt_0)$. By (ii), there exists a unique pair $(P, \alpha), (P, -\alpha) \in R^P$ of opposite roots such that the function $\chi_\alpha : \Xi_{(P,\delta)} \to \mathbb{C}^\times$ defined by $\chi_\alpha(\xi_t) := \alpha(t) - 1$ has the property that $M \cap U = \{\xi \mid \chi(\xi) = 0\} \cap U$. Observe that $\chi_\alpha(\mathfrak{s}_M(\xi_t)) = -\alpha(t)^{-1}\chi_\alpha(\xi_t)$. Let $U_T \subset T_u^P$ denote the open neighborhood of $e \in T_u^P$ such that $\xi_t \in U$ iff $t \in U_T$.

Suppose now that the order of the pole of c at M is larger than 1. Then for an arbitrary smooth section f with support in U as before, there exists a smooth section h with support in U such that $\chi_\alpha^{-2} f = ch$. Hence $p_{\mathcal{W}}(\chi_\alpha^{-2} f)$ is smooth by Theorem 5.3 of [9] (see (4.2)). In view of the choice of U and (4.2) this implies that the expression

$$\chi_\alpha^{-2}(\xi_t)(f(\xi_t) + \alpha(t)^2 \pi(\mathfrak{s}_M, \xi_t)^{-1} f(\mathfrak{s}_M(\xi_t))\pi(\mathfrak{s}_M, \xi_t)) \tag{4.3}$$

is smooth as a function of $t \in U_T$, for any choice of f. (Recall that $\pi(\mathfrak{s}_M, \xi)$ is smooth and invertible as a function of $\xi \in \Xi_u$, cf. Theorem 3.5) But if we choose f such that $f(\xi) = \mathrm{Id}_V$ we see that this is impossible. This proves (iii).

Let us finally prove (iv). We use the same set-up as above in the proof of (iii), but now with $\chi_\alpha^{-1} f = ch$ for some smooth section h. The equation (4.3) now becomes

$$\chi_\alpha^{-1}(\xi_t)(f(\xi_t) - \alpha(t)\pi(\mathfrak{s}_M, \xi_t)^{-1} f(\mathfrak{s}_M(\xi_t))\pi(\mathfrak{s}_M, \xi_t)), \tag{4.4}$$

and again we know that this should be smooth as a function of $t \in U_T$. This implies at $t = e$ that

$$f(\xi) - \pi(\mathfrak{s}_M, \xi)^{-1} f(\xi)\pi(\mathfrak{s}_M, \xi) = 0 \tag{4.5}$$

for all smooth sections f supported on U. But for any $A \in \mathrm{End}(V)$ there exists such a smooth section with $f(\xi) = A$, thus equation (4.5) implies that $\pi(\mathfrak{s}_M, \xi)$ is a scalar. $\square$

Definition 4.4. *Let $\xi \in \Xi_u$. We denote by $\mathcal{W}^m_{\xi,\xi} \subset \mathcal{W}_{\xi,\xi}$ the subgroup generated by the mirror reflections $\mathfrak{s}_M$ with $M \in \mathcal{M}$ such that $\xi \in \mathcal{M}$. The subgroupoid $\mathcal{W}^m$ whose set of objects is Ξ_u and whose set of arrows consists of the union of the sets $\mathcal{W}^m_{\xi,\xi}$ is a normal subgroupoid of $\mathcal{W}$.*

The statement that $\mathcal{W}^m$ is normal in $\mathcal{W}$ (i.e. invariant for conjugation in $\mathcal{W}$) follows immediately from the fact that $\mathcal{M}$ is $\mathcal{W}$-invariant.

The isotropy group $\mathcal{W}_{\xi,\xi}$ acts linearly on $\mathfrak{a}^P$ by identifying $\mathfrak{a}^P$ with the tangent space of $\Xi_{(P,\delta),u}$ at ξ via the local diffeomorphism

$$\mathfrak{a}^P \to \Xi_{(P,\delta),u} \tag{4.6}$$

$$x \to (\xi)_{\exp(2\pi i x)} \tag{4.7}$$

centered at ξ.

Definition 4.5. *Let $\xi \in \Xi_{(P,\delta),u}$ and consider the subset $R_\xi^{(P,\delta)} \subset R^P$ consisting of the roots (P, α) such that the zero set of the function χ_α defined by $\chi_\alpha(\xi_t) = \alpha(t) - 1$ is locally near ξ equal to a mirror $M_{(P,\alpha),\xi}$ containing ξ.*

Proposition 4.6. *The set $R_\xi^{(P,\delta)} \subset \mathfrak{a}^{P,*}$ is a reduced integral root system such that $\mathcal{W}^m_{\xi,\xi} \simeq W(R_\xi^{(P,\delta)})$.*

Proof. Recall the definitions of Appendix 7. The group $\mathcal{W}^m_{\xi,\xi}$ is by definition generated by the mirror reflections in the mirrors of the form $M_{(P,\alpha),\xi}$ with $(P,\alpha)\in R_\xi^{(P,\delta)}$, and it is clear that $R_\xi^{(P,\delta)}$ is invariant for $\mathcal{W}^m_{\xi,\xi}$. Therefore up to normalization we see that $R_\xi^{(P,\delta)}$ is the root system of the finite real reflection group $\mathcal{W}^m_{\xi,\xi}$ (in the sense of [4, Section 2.2]). Let us now consider the integrality of this root system. By Theorem 7.2, for all $(P,\tilde{\alpha})\in R_\xi^{(P,\delta)}$ the $\mathfrak{W}_P$-orbit of $M_{(P,\tilde{\alpha}),\xi}$ contains a mirror $M_{(Q,\gamma),\xi'}$ such that $(Q,H_\gamma):=\mathrm{Ker}(Q,\gamma)$ is the hyperplane in $\mathfrak{a}^Q$ associated to a simple root $\gamma\in F_0\backslash Q$. This implies that the $\mathfrak{W}_P$-orbit of $\tilde{\alpha}$ contains a root $\tilde{\gamma}=w\tilde{\alpha}\in R_{Q\cup\{\gamma\}}$ such that $\tilde{\gamma}\in\gamma+\mathbb{Z}R_Q$. If we put $\alpha=w^{-1}\gamma$ then we have $(P,H_{\tilde{\alpha}})=(P,H_\alpha)$, and moreover $P\cup\{\alpha\}$ is a system of simple roots for a (possibly non-standard) parabolic subsystem of roots. Moreover, from Theorem 4.3 we see that the elementary conjugation $\mathfrak{s}^P_{P'}$ of this parabolic root system leaves P fixed, i.e. P is self opposed in $P':=P\cup\{\alpha\}$ (see paragraph 7.0.4). Now let us fix a $\mathcal{W}^m_{\xi,\xi}$-invariant inner product in $\mathfrak{a}^P$. We need to show that $\langle\overline{\alpha},\overline{\beta}\rangle\in\mathbb{Z}$ holds for all $\overline{\alpha}:=(P,\alpha)\in R_\xi^{(P,\delta)}$ and $\overline{\beta}:=(P,\beta)\in R_\xi^{(P,\delta)}$. But since both α and β can be replaced by roots which form a simple system of roots together with P such that P is self opposed in these systems, the integrality assertion follows from the proof of Theorem 10.4.2 of [4]: Let $M=M_{(P,\alpha),\xi,u}$ then

$$\mathfrak{s}_M(\beta):=\mathfrak{s}_P^{P\cup\{\alpha\}}(\beta)=w_{P\cup\{\alpha\}}w_P(\beta)\in w_{P\cup\{\alpha\}}(\beta+\mathfrak{a}_P)=(\beta+\lambda\alpha)+\mathfrak{a}_P \tag{4.8}$$

with $\lambda\in\mathbb{Z}$ as desired. Recall that R^P consists only of *primitive* restrictions of roots of $R_0\backslash R_P$. Therefore it is now clear that $R_\xi^{(P,\delta)}$ is integral and reduced. □

Definition 4.7. *Let* $R_{\xi,+}^{(P,\delta)}=R_\xi^{(P,\delta)}\cap R_+^P$, *and let* $\mathfrak{a}_\xi^{P,+}\subset\mathfrak{a}^P$ *be the positive Weyl chamber of* $R_{\xi,+}^{(P,\delta)}$. *We define*

$$\mathfrak{R}_\xi=\{w\in\mathcal{W}_{\xi,\xi}\mid w(\mathfrak{a}_\xi^{P,+})=\mathfrak{a}_\xi^{P,+}\}. \tag{4.9}$$

Proposition 4.8. *The subgroup* $\mathfrak{R}_\xi\subset\mathcal{W}_{\xi,\xi}$ *is a complement for the normal subgroup* $\mathcal{W}^m_{\xi,\xi}$. *Hence*

$$\mathcal{W}_{\xi,\xi}=\mathfrak{R}_\xi\ltimes\mathcal{W}^m_{\xi,\xi}. \tag{4.10}$$

Proof. The group $\mathcal{W}_{\xi,\xi}$ preserves the set $R_\xi^{(P,\delta)}$ of roots of the finite reflection group $\mathcal{W}^m_{\xi,\xi}$, and thus the choice of a positive Weyl chamber induces a splitting of $\mathcal{W}_{\xi,\xi}$ as indicated. □

5. The Knapp-Stein linear independence theorem

In this section we will prove the Knapp-Stein linear independence in the present context of affine Hecke algebras [14], [30]).

Let $\eta=(P,\delta,t)\in\Xi_{(P,\delta),u}$ be an R_P-generic (cf. [9, Definition 2.5]) induction parameter in a small open $\mathcal{W}_{\xi,\xi}$-invariant neighborhood U of $\xi\in\Xi_{(P,\delta),u}$. Let W_Pr denote the central character of δ. We will need to use Lusztig's first reduction theorem, in the version as discussed in [9]; we refer the reader to [9, Section 2.6] and [16] for further details. The reduction theorem describes the structure of the formal completion of $\mathcal{H}$ at the central character $W_0(rt)$, as a matrix algebra with coefficients in the formal completion at $\omega_t=W_P(rt)=tW_Pr$ of the Levi subalgebra $\mathcal{H}^P$. The orbit $W_0(rt)$ is partitioned in equivalence classes of the form $w\omega_t$

with $w \in W^P$. For each equivalence class $w\omega_t$ there exits an idempotent $e_{w\omega_t}$ in the formal completion $\bar{\mathcal{H}}_{W_0(rt)}$ of $\mathcal{H}$ at the central character $W_0(rt)$. These idempotents form a complete orthogonal set of idempotents in $\bar{\mathcal{H}}_{W_0(rt)}$. In the present context of R_P-generic induction parameters the reduction theorem asserts that $e_{\omega_t}\bar{\mathcal{H}}_{W_0(rt)}e_{\omega_t} = e_{\omega_t}\bar{\mathcal{H}}^P_{\omega_t}$ where e_{ω_t} is a central idempotent on the right hand side, and that we have a decomposition

$$\bar{\mathcal{H}}_{W_0(rt)} = \bigoplus_{u,v\in W^P} \iota^0_u e_{\omega_t}\bar{\mathcal{H}}^P_{\omega_t}\iota^0_{v^{-1}} \tag{5.1}$$

which yields an isomorphism of $\bar{\mathcal{H}}_{W_0(rt)}$ and a matrix algebra of size $N = |W^P|$ and coefficients in $e_{\omega_t}\bar{\mathcal{H}}^P_{\omega_t}$. The theorem moreover asserts in this situation that if $w(P) = Q \in \mathcal{P}$ then the conjugation map $c_w : x \to \iota^0_w x \iota^0_{w^{-1}}$ is well defined on $e_{\omega_t}\bar{\mathcal{H}}^P_{\omega_t} \subset \bar{\mathcal{H}}_{W_0(rt)}$ and defines an algebra isomorphism

$$c_{\iota^0_w} : e_{\omega_t}\bar{\mathcal{H}}^P_{\omega_t} \xrightarrow{\sim} e_{w(\omega_t)}\bar{\mathcal{H}}^Q_{w(\omega_t)} \tag{5.2}$$

which coincides with the isomorphism originating from the isomorphism of root data $\mathcal{R}^P \xrightarrow{\sim} \mathcal{R}^Q$ induced by w.

We also use the concept of the *constant term* of (matrix coefficients of) tempered representations along a standard parabolic subset $P \in \mathcal{P}$, see [9, Section 3.6] and [9, Section 6]. The subset of $w \in W^P$ such that $e_{w\omega_t}$ contributes to the constant term V^P_η along P of the tempered module V_η is equal to $W_{P,P}$ (cf. [9, Proposition 6.12], where we remark that $W_{P,P} = D^{P,P}$ in the situation $Q = P$). By the Morita equivalence Proposition 3.9 we see that $\mathcal{W}_{\xi,\xi} \simeq \mathfrak{W}_{\mathcal{K}\xi,\mathcal{K}\xi} \subset W_{P,P}$. We will identify $\mathcal{W}_{\xi,\xi}$ with this subgroup of $W_{P,P}$ in the rest of this section.

Choose a complete set S of representatives for the left cosets of $\mathcal{W}_{\xi,\xi}$ in $\mathcal{W}_{P,P}$. For each $s \in S$ and $\mathfrak{r} \in \mathfrak{R}_\xi$ we define

$$E_{s,\mathfrak{r};t} = \sum_{w\in\mathcal{W}^m_{\xi,\xi}} e_{sw\mathfrak{r}\omega_t} \in \bar{\mathcal{H}}_{W_0(rt)}, \tag{5.3}$$

which is an idempotent of the formal completion of $\mathcal{H}$ at the central character $W_0(rt)$. Recall that rt is R_P-generic.

Proposition 5.1. *For all $s \in S$, $\mathfrak{r} \in \mathfrak{R}_\xi$ and $\eta \in U$ we define a projection $p(s,\mathfrak{r},\eta)$ in $V_\eta = i(V_\delta)$ by $p(s,\mathfrak{r},\eta) := \pi(\eta, E_{s,\mathfrak{r};t})$.*

- (i) *Viewed as rational function of η, $p(s,\mathfrak{r},\eta)$ is regular for $\eta \in U$.*
- (ii) *For all $\eta \in U$, $\sum_{s\in S,\mathfrak{r}\in\mathfrak{R}_\xi} p(s,\mathfrak{r},\eta)$ is the projection onto V^P_η, the constant part of V_η along P (see* [9, Section 3.6] *for the definition of the constant part of a tempered module).*
- (iii) *The collection of idempotents $\{p(s,\mathfrak{r},\eta)\}$ is mutually orthogonal in* $\mathrm{End}(i(V_\delta))$ *for all $\eta \in U$.*
- (iv) *For all $s,\mathfrak{r}$, and all $\eta \in U$: $p(s,\mathfrak{r},\eta)$ is an endomorphism of the $\mathcal{H}^P$-module structure on $i(V_\delta)$ obtained by restricting $\pi(\eta)$ to $\mathcal{H}^P$.*

Proof. For generic η the properties (ii), (iii) and (iv) follow straightforward from the definitions and from [9], Sections 3.6, 6.1, 6.2 (especially Corollary 6.9, in which one should observe that $\mathcal{W}_{P,P} = \{d \in D^{P,P} \mid d(P) = P\}$) and 6.3. From this remark it is clear that (i) implies (iii), (iv). But [9, Proposition 7.8] implies that the projection $V_\eta \to V^P_\eta$ is also smooth in

$\eta \in \Xi_{(P,\delta),u}$, and thus also (ii) will follow from the generic case provided we know (i). Thus it remains only to prove (i).

It is obviously enough to consider the case $s = e$ and $\mathfrak{r} = e$ (replace ξ by $s\xi$ and t by $s\mathfrak{r}t$ in (5.3)). We compute a matrix coefficient of $P(\eta) := p(e, e, \eta)$. Let $a, b \in i(V_\delta)$. Then, using the notations of [9, Subsection 6.2, 6.3 and 6.4] we have:

$$\begin{aligned}\langle a, P(\eta)b\rangle &= f_{a,P(\eta)b}(\eta, 1)\\ &= \sum_{d\in\mathcal{W}^m_{\xi,\xi}} f^d_{a,b}(\eta, 1)\\ &= \sum_{d\in\mathcal{W}^m_{\xi,\xi}} f^1_{\pi(d,\eta)a,\pi(d,\eta)b}(d\eta, 1)\\ &= \sum_{d\in\mathcal{W}^m_{\xi,\xi}} f^1_{a,b}(d\eta, 1)\\ &= \sum_{d\in\mathcal{W}^m_{\xi,\xi}} c(d\eta)(c(d\eta)^{-1} f^1_{a,b}(d\eta, 1)).\end{aligned} \tag{5.4}$$

Here the first two equalities follow form a direct unwinding of definitions, the third equality follows from an application of [9, Lemma 6.14], the fourth is the unitarity property [25, Theorem 4.33] of the intertwiners $\pi(d, \eta)$, and the last one is trivial. By [9, Theorem 6.18] the expression $c(d\eta)^{-1} f^1_{a,b}(d\eta, 1)$ is regular for η in a small tubular neighborhood of $\Xi_{(P,\delta),u}$. By Theorem 6.3 (iii), the singularities of $c(d\eta)$ for $d \in \mathcal{W}^m_{\xi,\xi}$ and for $\eta \in U$ are poles of order at most one along the mirrors M of $\mathcal{W}^m_{\xi,\xi}$ which contain ξ. On the other hand, the expression on the right hand side of the last equality of equation (5.4) also shows that $\langle a, P(\eta)b\rangle$ is $\mathcal{W}^m_{\xi,\xi}$-invariant as a function of η. The product of the function $U \ni \eta \to \langle a, P(\eta)b\rangle$ by

$$\pi := \prod_{(P,\alpha)\in R^{(P,\delta)}_{\xi,+}} (P, \alpha) \tag{5.5}$$

extends to a $\mathcal{W}^m_{\xi,\xi}$-skew invariant analytic function on U. It is a well known basic fact from the invariant theory of finite real reflection groups that a $\mathcal{W}^m_{\xi,\xi}$-skew invariant analytic function on U is divisible by π. This implies that the apparent first order poles of $\langle a, P(\eta)b\rangle$ along the mirrors of $\mathcal{W}^m_{\xi,\xi}$ are removable themselves. Therefore $\langle a, P(\eta)b\rangle$ extends to an analytic function of $\eta \in U$. □

Corollary 5.2. (i) *For all $\eta \in U$ we have a decomposition*

$$V^P_\eta = \bigoplus_{s\in S,\mathfrak{r}\in\mathfrak{R}_\xi} V^P_{s,\mathfrak{r},\eta} \tag{5.6}$$

of the $\mathcal{H}^P$-module V^P_η as a direct sum of $\mathcal{H}^P$-submodules $V^P_{s,\mathfrak{r},\eta}$ defined by $V^P_{s,\mathfrak{r},\eta} := p(s, \mathfrak{r}, \eta)(V_\eta)$.

(ii) *For all $s \in S, \mathfrak{r} \in \mathfrak{R}_\xi$, all the irreducible subquotients of the finite length $\mathcal{H}^P$-module $V^P_{s,\mathfrak{r},\xi}$ are isomorphic to $(\delta^s)_{st^0}$.*

Proof. (i) This is a direct consequence of Proposition 5.1.

(ii) Assume that t is such that $\eta = (P, \delta, t) \in U$ is generic. We have according to [9, equation (3.6)] that

$$i(V_\delta) \simeq \bigoplus_{u \in W^P} \iota_u^0 e_{\omega_t} \otimes V_{\delta_t}. \tag{5.7}$$

From (5.3) and the orthogonality of the idempotents we then conclude that

$$V^P_{s,\mathfrak{r},\eta} = \bigoplus_{w \in \mathcal{W}^m_{\xi,\xi}} e_{sw\mathfrak{r}\omega_t} \iota^0_{sw\mathfrak{r}} \otimes V_{\delta_t} \tag{5.8}$$

Using the definition of the normalized intertwining operators (see Remark 3.8 and [9, equation (3.7)]) we see that this is isomorphic to

$$V^P_{s,\mathfrak{r},\eta} = \bigoplus_{w \in \mathcal{W}^m_{\xi,\xi}} \pi(\mathfrak{r}^{-1} w^{-1} s^{-1}, sw\mathfrak{r}(\eta))(e_{\omega_{sw\mathfrak{r}t}} \otimes V_{\delta^s_{sw\mathfrak{r}t}}), \tag{5.9}$$

so that the $\mathcal{H}^P$-module $V^P_{s,\mathfrak{r},\eta}$ is isomorphic to a direct sum of the irreducible modules $\mathcal{H}^P$-modules $(\delta^s)_{sw\mathfrak{r}t}$, where w runs over all the elements of $\mathcal{W}^m_{\xi,\xi}$. If we substitute $t = t_0$ (this corresponds to taking $\eta = \xi$) then each of these irreducible summands coincides with $(\delta^s)_{st^0}$. Hence the character of $V^P_{s,\mathfrak{r},\xi}$ (since $V^P_{s,\mathfrak{r},\eta}$ depends smoothly on η) is simply $|\mathcal{W}^m_{\xi,\xi}|$ times the character of $(\delta^s)_{st^0}$. Therefore all the irreducible subquotients of the finite length $\mathcal{H}^P$-module $V^P_{s,\mathfrak{r},\xi}$ are isomorphic to $(\delta^s)_{st^0}$. □

Corollary 5.3. *The $\mathcal{H}^P$-module $V^P_{s,\mathfrak{r},\xi}$ has a unique irreducible submodule, which is isomorphic to $(\delta^s)_{st^0}$ for all $\mathfrak{r} \in \mathfrak{R}_\xi$ and all $s \in S$.*

Proof. By symmetry it suffices to prove this for $s = e$.

By Frobenius reciprocity [9, Proposition 3.18] we have:

$$\mathrm{End}_{\mathcal{H}}(\pi(\xi)) = \mathrm{Hom}_{\mathcal{H}^P}(\delta_{t_0}, V^P_\xi). \tag{5.10}$$

By [9, Corollary 5.4] we know that $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$ is the complex linear span of the operators $\pi(g, \xi)$ with $g \in \mathcal{W}_{\xi,\xi}$. We have seen that $\pi(g, \xi)$ is a scalar for $g \in \mathcal{W}^m_{\xi,\xi}$. Thus $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$ is already spanned by the operators $\pi(\mathfrak{r}, \xi)$ with $\mathfrak{r} \in \mathfrak{R}_\xi$. Hence the dimension of $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$ is at most $|\mathfrak{R}_\xi|$.

On the other hand, Corollary 5.2 implies that $(\delta)_{t^0}$ occurs at least once as a submodule of $V^P_{e,\mathfrak{r},\xi}$, for every $\mathfrak{r} \in \mathfrak{R}_\xi$. Combining this with the Frobenius reciprocity formula (5.10) we obtain that $(\delta)_{t^0}$ occurs precisely once as a submodule of $V^P_{s,\mathfrak{r},\xi}$ for every $\mathfrak{r} \in \mathfrak{R}_\xi$. Again invoking Corollary 5.2 we conclude that this irreducible submodule is in fact the unique irreducible submodule of $V^P_{s,\mathfrak{r},\xi}$. □

Theorem 5.4. *For all $\xi \in \Xi_u$, we have*

$$\mathrm{End}_{\mathcal{H}}(\pi(\xi)) = \sum_{\mathfrak{r} \in \mathfrak{R}_\xi} \mathbb{C}\pi(\mathfrak{r}, \xi), \tag{5.11}$$

the complex linear span of the operators $\pi(r, \xi)$ with $r \in \mathfrak{R}_\xi$. Moreover, these operators are linearly independent, so that

$$\dim \mathrm{End}_{\mathcal{H}}(\pi(\xi)) = |\mathfrak{R}_\xi|. \tag{5.12}$$

Proof. In the course of the proof of Corollary 5.3 it was shown that the dimension of $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$ is in fact precisely equal to $|\mathfrak{R}_\xi|$. It was also remarked in the proof of Corollary 5.3 that the space of endomorphisms of $\pi(\xi)$ is spanned by the operators $\pi(\mathfrak{r}, \xi)$ with $\mathfrak{r} \in \mathfrak{R}_\xi$. □

Theorem 5.5. *Let $\xi \in \Xi_u$ be a standard induction datum for $\mathcal{H}$.*

(i) *Let γ_ξ denote the restriction of the 2-cocycle $\gamma_{\mathcal{W},\Xi}$ to $\mathcal{W}_{\xi,\xi}$. This 2-cocycle is cohomologous to the pull-back of a 2-cocycle on $\mathfrak{R}_\xi$ (which we also denote by γ_ξ by abuse of notation) via the natural projection $\mathcal{W}_{\xi,\xi} \to \mathfrak{R}_\xi$.*

(ii) *The map $\mathfrak{R}_\xi \ni \mathfrak{r} \to \pi(\mathfrak{r}, \xi)$ extends by linearity to an algebra isormorphism $\pi(\cdot, \xi)$ from the γ_ξ-twisted complex group algebra ${}^\gamma\mathbb{C}[\mathfrak{R}_\xi]$ of $\mathfrak{R}_\xi$ to the commutant algebra* $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$.

(iii) *Up to the choice of the isomorphism $\pi(\cdot, \xi)$ (which depends on our choice of normalized intertwining operators (3.7)) there exists a unique bijection*

$$\widehat{{}^\gamma\mathbb{C}[\mathfrak{R}_\xi]} \to \hat{\mathcal{S}}_{\mathcal{W}\xi} \qquad (5.13)$$
$$\rho \to \pi_\rho$$

(where $\hat{\mathcal{S}}_{\mathcal{W}\xi}$ denotes a complete set of representatives of the finite set of isomorphism classes of irreducible representations of $\mathcal{S}$ with central character $\mathcal{W}\xi$ (central character in the sense of [9, Corollary 5.5]*)) such that we have a decomposition (recall that V_ξ is the vector space on which $\pi(\xi)$ is realized)*

$$V_\xi = \bigoplus_\rho \pi_\rho \otimes \rho \qquad (5.14)$$

as a $\mathcal{S} - {}^\gamma\mathbb{C}[\mathfrak{R}_\xi]^{op}$-bimodule. Here the sum runs over $\rho \in \widehat{{}^\gamma\mathbb{C}[\mathfrak{R}_\xi]}$ (viewed as irreducible right ${}^\gamma\mathbb{C}[\mathfrak{R}_\xi]^{op}$-module).

Proof. The first claim (i) comes from the fact that the projective representation $\mathbb{P}\pi(\xi)$ on $\mathbb{P}(V_\xi)$ descends to $\mathfrak{R}_\xi$ by Theorem 4.3(iv). Hence $\pi(\mathfrak{r}, \xi) \in \mathrm{U}(V_\xi)$ is a lifting of $\mathbb{P}\pi(w^0\mathfrak{r}, \xi) \in \mathbb{P}\,\mathrm{U}(V_\xi)$ for all $w^0 \in \mathcal{W}^0_{\xi,\xi}$. Therefore $[\gamma_\xi]$ descends to $\mathfrak{R}_\xi \times \mathfrak{R}_\xi$. The remaining part of the Theorem follows from Theorem 5.4 by the arguments as in [1, p. 87-88]. □

6. The cocycle γ for classical Hecke algebras

In this section we prove the triviality of the 2-cocycles $\gamma_{\mathcal{W},\Xi_u}$ for the classical Hecke algebras. The computation is based on the classification [26] of the discrete series representations of classical affine Hecke algebras.

Theorem 6.1. *Let $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ be an irreducible root datum of classical type, and let q be an arbitrary positive parameter function for $\mathcal{R}$.*

(i) *We have $[\gamma_\xi] = 1$ for all $\xi \in \Xi_u$.*

(ii) *We have $[\gamma_{\mathcal{W},\Delta}] = 1$ if R_0 is not of type D_n with $n > 8$, or if X is not the root lattice of R_0.*

Proof. We use that the assertion $[\gamma_{\mathcal{W},\Delta}] = 1$ of (ii) implies (i) by taking the pull-back to $\mathcal{W}_{\Xi_u}$ (using Definition 3.8). Hence in all cases mentioned in (ii) it suffices to prove (ii). This is an aggregate of various special cases which are treated separately below. In the remaining case $R_0 = \mathrm{D}_n$ with $n > 8$ and X the root lattice of R_0 we will show directly that (i) holds. □

Remark 6.2. *Remarkably, if R_0 is of type* D_n *with* $n > 8$ *and* X *equals the root lattice of* R_0 *then there exist discrete series representations* δ_P *of* $\mathcal{H}_P$ *such that* $[\gamma_{(P,\delta_P)}] \neq 1$.

We will not give the proof here of the nontriviality of $[\gamma_{(P,\delta_P)}]$ in this exceptional case, but for a precise formulation see paragraph 6.7.2. Using the description of the discrete series for affine Hecke algebras of type D_n as given in [26], the proof consists of tracing the action of $\mathcal{W}_{\xi,\xi}$ on the isomorphisms constructed in Lusztig's first reduction theorem [16].

6.1. **A remark on isogenous affine Hecke algebras**

For later use we list the following useful general fact.

Lemma 6.3. *Let* $\mathcal{H}$ *be a semisimple affine Hecke algebra. Let* $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ *be the based root datum of* $\mathcal{H}$, *and* q *its parameter function. Let* $\mathcal{H}^e$ *be an isogenous extension of* $\mathcal{H}$, *i.e. a semisimple affine Hecke algebra with root datum* $\mathcal{R}^e = (X^e, Y^e, R_0, R_0^\vee, F_0)$ *where* $X \subset X^e$ *is an extension of lattices such that* $R_{nr} = R^e_{nr}$, *and such that the parameter* q^e *of* $\mathcal{H}^e$ *is equal to* q, *viewed as functions on* $R_{nr}^\vee$. *Then* $\mathcal{H} \subset \mathcal{H}^e$ *is an isometric embedding of* $\mathcal{H}$ *as a* $*$*-subalgebra of* $\mathcal{H}^e$ *of finite index, and the induction and restriction functors between* $\mathrm{Rep}(\mathcal{H})$ *and* $\mathrm{Rep}(\mathcal{H}^e)$ *send irreducible unitary (resp. discrete series) representations to finite direct sums of irreducible unitary (resp. discrete series) representations.*

Proof. Let $\Lambda_r \subset X$ be the root lattice of R_0. We introduce the finite abelian groups $\Omega = X/\Lambda_r$ and $\Omega^e = X^e/\Lambda_r$, and we denote by $\mathcal{H}_r \subset \mathcal{H}^e$ the affine Hecke algebra with root system R_0 (and basis F_0) and whose root datum has the root lattice Λ_r of R_0 as lattice X. As is well known, we have $\mathcal{H}^e = \mathcal{H}_r \rtimes \Omega^e$ and $\mathcal{H} = \mathcal{H}_r \rtimes \Omega \subset \mathcal{H}^e$. From this and the standard description of the Hilbert algebra structures ($*$ and the trace τ) on $\mathcal{H}$ and $\mathcal{H}^e$ we see that $\mathcal{H} \subset \mathcal{H}^e$ is an isometric embedding of a $*$-subalgebra. It is immediate from this that the restriction functor preserves unitarity and discreteness.

Now let us consider induction. Multiplication yields an action of Ω^e on $\mathcal{H}^e$ which permutes the standard orthonormal basis elements $\{N_w\}_{w\in W^e_P}$ of $\mathcal{H}^e$ freely (with W^e the extended affine Weyl group with root system R_0). Choose a set $\omega_1, \dots, \omega_n$ of representatives for the cosets of Ω in Ω^e. Then we have an orthogonal decomposition $\mathcal{H}^e = \omega_1\mathcal{H} \oplus \cdots \oplus \omega_n\mathcal{H}$. For all $h \in \mathcal{H}$ we have $\omega_i h = h^{\omega_i}\omega_i$, where $h \to h^{\omega_i}$ is a (special) affine diagram automorphism associated with ω_i of $\mathcal{H}$. Such automorphisms are isometries since they permute the standard orthonormal basis of $\mathcal{H}$. Let (V, π) be a finite dimensional representation of $\mathcal{H}$. The underlying vector space $i(V) = \mathcal{H}^e \otimes V$ of the induced representation $i(\pi)$ is the direct sum of the subspaces $V_i := \omega_i^{-1} \otimes V$, and we identify each V_i with V by the map $V \ni v \mapsto \omega_i^{-1} \otimes v \in V_i$. If (V, π) is unitary then $(i(V), i(\pi))$ unitary with respect to the Hilbert space structure on $i(V)$ which is the orthogonal direct sum of the subspaces V_i (each equipped with the transfer of the Hilbert structure of V under this identification). We see that the character $\chi_{i(\pi)}$ satisfies $\chi_{i(\pi)}(\omega_i h) = \chi_\pi(h^{\omega_i})$ for each i. With the above orthogonal decomposition of $\mathcal{H}^e$ and the fact that $h \to h^{\omega_i}$ is an isometry of $\mathcal{H}$ we see that $i(\pi)$ is a discrete series character if and only if π is a discrete series character. □

6.2. R_0 **has only irreducible components of type A**

In this situation we prove a more general result:

Proposition 6.4. *Let R_0 be a root system whose irreducible components are all of type* A*, and let $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ be an arbitrary (not necessarily semisimple) root datum whose underlying root system is R_0. Then $\gamma := \gamma_{\mathcal{W},\Delta} = 1$.*

Proof. If R_0 has only irreducible components of type A then the same is true for any of its standard parabolic subsystems $R_P \subset R_0$. Let $\mathcal{H}_P^e := \mathcal{H}_P(X^e, q^e)$ be the extended semisimple affine Hecke algebra whose root datum has underlying root system R_P with basis $F_P \subset F_0$ and whose lattice $X_P^e = \Lambda_P$ is the weight lattice of R_P. Then $\mathcal{H}_P^e$ is a tensor product of various extended affine Hecke algebras $\mathcal{H}_{\lambda_i-1}^{A,e} := \mathcal{H}_{\lambda_i-1}^{A}(\Lambda_{\lambda_i-1}^{A}, q)$ of type A_{λ_i-1} (for various $\lambda_i \geq 2$). Since it is well known [33] that the irreducible discrete series representations of $\mathcal{H}_{\lambda_i-1}^{A,e}$ all have dimension 1, it follows from the above that all the discrete series representations δ_P^e of $\mathcal{H}_P^e$ are one dimensional. In particular, the restriction of an irreducible discrete series representation of $\mathcal{H}_P^e$ to $\mathcal{H}_P$ is irreducible. By Lemma 6.3 and by the adjointness of restriction and induction we see that all irreducible discrete series representations of $\mathcal{H}_P$ are obtained in this way, and in particular are one dimensional. The projective representation $\tilde{\delta}$ of $\mathcal{W}_\Delta$ is thus one dimensional, hence trivial. In particular, its class $[\gamma_{\mathcal{W},\Delta}]$ is trivial. □

6.3. R_0 of type $\mathbf{B}_n$

The next result generalizes a result of Slooten [31].

Proposition 6.5. *Suppose that $\mathcal{R}$ is of type $\mathrm{C}_n^{(1)}$ (i.e. R_0 is of type B_n and X is the root lattice of R_0) with arbitrary positive parameter function $q = (q_0, q_1, q_2)$, where (using the standard realization for $C_n^{(1)}$) $q_0 = q(s_{2x_n})$, $q_1 = q(s_{x_i - x_{i+1}})$ and $q_2 = q(s_{1-2x_1})$. Then $[\gamma_{W,\Delta}] = 1$.*

Proof. In order to analyze the cocycle $\gamma_{\mathcal{W},\Delta}$, let us first look more carefully at the type A case. Let $\mathcal{H}_{n-1}^A(X, q)$ be an affine Hecke algebra with R_0 of type A_{n-1} and with lattice X (situated between the root lattice Λ_r and the weight lattice Λ of R_0) and parameter $q \neq 1$ (if $q = 1$ there are no discrete series). If $q \neq 1$ then the set $\Delta_{n-1}^A(X, q)$ of equivalence classes of discrete series representations of the affine Hecke algebra of type A_{n-1} with lattice X is in canonical bijection with the set $K_{n-1}^A(X)$ of characters of X which are trivial on the root lattice Λ_r of R_0, through the central character map. Namely, in terms of the notation of [26, Section 8] we have $K_{n-1}^A(X) = \Gamma^{max}/\Gamma$ where $\Gamma^{max} = \mathrm{Hom}(\Lambda/\Lambda_r, \mathbb{C}^\times) \approx C_n$. The group Γ^{max} acts simply transitively on the set of vertices $E(C^\vee)$ (notations as in loc. cit.). Hence the group K acts simply transitively on the set of Γ-orbits on $E(C^\vee)$, and for each $s \in E(C^\vee)$ we have $\Gamma_s = 1$. By [26, Theorem 8.7] the set Δ_{n-1}^A is a disjoint union over all Γ-orbits of $E(C^\vee)$ of the set of discrete series characters of the graded affine Hecke algebra $\mathbf{H}(R_{s(e),1}, V, F_{s(e),1}, k_e)$. In the type A_{n-1}-case, these are all isomorphic to a graded affine Hecke algebra of type A_{n-1}, and hence each of these contributes precisely one discrete series character (since we assume that $q \neq 1$, implying that $k_e = k \neq 0$). If $k \in K_{n-1}^A(X)$ we denote by δ_k the unique, one-dimensional discrete series character of the type A_{n-1} affine Hecke algebra $\mathcal{H}_{n-1}^A(X, q)$ whose central character has unitary part k. Then $\Delta_{n-1}^A(X) = \{\delta_k \mid k \in K_{n-1}^A(X)\}$. Through the twisting automorphisms ψ_k (see paragraph 3.2.2) the group $K_{n-1}^A(X)$ acts on $\Delta_{n-1}^A(X)$. We have

$$\delta_{k'}^k := \delta_{k'} \circ \psi_k^{-1} = \delta_{k'k^{-1}} \tag{6.1}$$

Now we return to the case where $\mathcal{R}$ is equal to $C_n^{(1)}$. The possible pairs (P, δ_P) with $\delta_P \in \Delta_P$ can be described explicitly as follows. First notice that $P \subset F_0$ is of type

$$\mathrm{A}_{\lambda_1-1} \times \mathrm{A}_{\lambda_2-1} \times \cdots \times \mathrm{A}_{\lambda_r-1} \times \mathrm{B}_l \tag{6.2}$$

where $l \leq n$ and where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a composition of $n-l$. In this situation $\mathcal{H}_P$ is of the form

$$\mathcal{H}^A_{\lambda_1-1}(\Lambda^A_{\lambda_1-1}, q_1) \otimes \mathcal{H}^A_{\lambda_2-1}(\Lambda^A_{\lambda_2-1}, q_1) \otimes \cdots \otimes \mathcal{H}^A_{\lambda_r-1}(\Lambda^A_{\lambda_r-1}, q_1) \otimes \mathcal{H}^B_l(q) \tag{6.3}$$

where $\mathcal{H}^A_{\lambda_i-1}(\Lambda^A_{\lambda_i-1}, q_1)$ denotes the extended affine Hecke algebra of type A_{λ_i-1} whose lattice equals the weight lattice $\Lambda^A_{\lambda_i-1}$ of A_{λ_i-1} and with parameter q_1, and where $H^B_l(q)$ is the affine Hecke algebra of type $\mathrm{C}^{(1)}_l$ with parameter q. Thus

$$\delta_P = \delta_{1,k_1} \otimes \delta_{2,k_2} \otimes \cdots \otimes \delta_{r,k_r} \otimes \delta \tag{6.4}$$

where $k_i \in K^A_{\lambda_i-1}(\Lambda^A_{\lambda_i-1})$ (a cyclic group of order λ_i), δ_i is the unique irreducible discrete series representation of $\mathcal{H}^A_{\lambda_i-1}(\Lambda^A_{\lambda_i-1}, q_1)$ with real infinitesimal character, and where δ is a discrete series representation of $\mathcal{H}^B_l(q)$. As discussed in paragraph 3.2.2, an element $g = k \times u \in \mathcal{W}_{P,Q} = K_Q \times \mathfrak{W}_{P,Q}$ gives rise to an automorphism $\psi_g : \mathcal{H}_P \to \mathcal{H}_Q$. In the present situation, u is a composition of a permutation of tensor legs of the form $\mathcal{H}^A_{\lambda_i-1}(\Lambda^A_{\lambda_i-1}, q_1)$ in (6.3) with equal λ_i, with a tensor product of automorphisms of the tensor factors $\mathcal{H}^A_{\lambda_i-1}(\Lambda^A_{\lambda_i-1}, q_1)$ induced by a diagram automorphisms of the finite Dynkin diagram of type A_{λ_i-1}. Recall that $K_Q = T^Q \cap T_Q \approx \mathrm{Hom}(X_Q/(X \cap \mathbb{R}Q), \mathbb{C}^\times)$ (by (3.3)). Therefore K_Q is the direct product of the cyclic groups $K^A_{\lambda_i-1}(\Lambda^A_{\lambda_i-1})$ of order λ_i, and $k = k_i \times k_2 \times \cdots \times k_r$ acts by the tensor product of the automorphisms ψ_{k_i} described above. The crucial observation to make here is that the action of ψ_g on the last tensor leg is always trivial. If we choose a basis vector for each of the one dimensional representations δ_{i,k_i} we obtain a natural identification of the vector space of δ_P with the vector space V_δ on which δ is realized. With this identification, we can choose $\psi_g = \mathrm{Id}_{V_\delta}$ for all g, and hence $\gamma_{\mathcal{W},\Delta} = 1$. □

Proposition 6.6. *Consider the irreducible* extended *affine Hecke algebra* $\mathcal{H} := \mathcal{H}^{B,e}_n$ *with* R_0 *of type* B_n *and* $X = \Lambda^B_n$*, the weight lattice of type* B_n*. Then* $[\gamma_{\mathcal{W},\Delta}] = 1$.

Proof. The proof of Proposition 6.5 changes only slightly. In the present situation q is restricted to the cases where $q_0 = q_2$. The last tensor leg of the algebra $\mathcal{H}_P$ in (6.3) changes to the extended algebra $\mathcal{H}^{B,e}_l$. For future reference we note that $\mathcal{H}_P$ is again of the form $\mathcal{H}_P = \mathcal{H}^{A,e}_P \otimes \mathcal{H}^{B,e}_P$, a tensor product of a number of extended type A Hecke algebras with an extended type B Hecke algebra (recall that the lattice $X_P = \Lambda_P$ underlying $\mathcal{H}_P$ is the projection of the weight lattice Λ onto the vector space $\mathbb{R}P$ which is indeed the weight lattice of R_P). Accordingly we have $\delta_P = \delta^{A,e}_P \otimes \delta^{B,e}_P$ (analogous to (6.4)).

The group $K^{B,\Lambda}_P = T_P \cap T^P$ that needs to be considered in the definition of $\mathcal{W}_\Delta$ equals, as before, the group of characters of Λ_P which are trivial on the sublattice $\Lambda \cap \mathbb{R}P$. In the present situation one checks easily that if there exists at least one odd λ_i then $\Lambda \cap \mathbb{R}P = \Lambda_{P,r}$, the root lattice of R_P. Therefore $K^{B,\Lambda}_P = K_P \times C_2$ in this case, where the group K_P is defined as above for the previous case $\mathcal{H}^B_n := \mathcal{H}(\mathrm{C}^{(1)}_n, q)$ (i.e. a direct product of cyclic groups) and where the extra factor C_2 (the group with 2 elements) acts (by twisting automorphisms) on rightmost tensor factor $\mathcal{H}^{B,e}_P$ only.

On the other hand, if all λ_i are even, then $\Lambda \cap \mathbb{R}P = \Lambda_{P,r} + (v + \Lambda_{P,r})$ where $v \in \Lambda_P$ is a vector having coordinates $\pm 1/2$ such that for each part λ_i of λ the corresponding coordinates of v sum up to zero (which is possible because λ_i is even). Hence in this case the group $K_P^{B,\Lambda}$ is equal to the kernel of the unique quadratic character ρ of $K_P \times C_2$ which is nontrivial on all factors (recall that all factors are even cyclic groups, hence admit a unique nontrivial quadratic character). Thus $K_P^{B,\Lambda}$ is a subgroup of index 2 in $K_P \times C_2$ in this case.

It is at this point useful to remark that $\mathcal{W}_\Delta$ is equivalent to a finite union of the isotropy groups $\mathcal{W}_{(P,\delta_P),(P,\delta_P)}$ by choosing a complete set of representatives for the $\mathcal{W}_\Delta$-orbits of pairs (P,δ_P). Hence it is enough to show that the restriction γ_{P,δ_P} of γ to $\mathcal{W}_{(P,\delta_P),(P,\delta_P)}$ is trivial for each pair (P,δ_P). Recall that $\delta_P = \delta_P^{A,e} \otimes \delta_P^{B,e}$. By the above we see that $\mathcal{W}_{(P,\delta_P),(P,\delta_P)} \subset \mathcal{W}^e_{(P,\delta_P),(P,\delta_P)} := \mathcal{W}^{A,e}_{(P,\delta_P^{A,e}),(P,\delta_P^{A,e})} \times \mathcal{W}^{B,e}_{(P,\delta_P^{B,e}),(P,\delta_P^{B,e})}$ (a subgroup of index at most 2, depending on P), where the factor $\mathcal{W}^{B,e}_{(P,\delta_P^{B,e}),(P,\delta_P^{B,e})} \subset K_l^B(\Lambda_l^B) = C_2$ (recall that the finite Dynkin diagram of type B_l has no nontrivial diagram automorphisms) is either trivial or isomorphic to C_2 (depending on $\delta_P^{B,e}$). The projective representation $\tilde{\delta}_P$ of $\mathcal{W}_{(P,\delta_P),(P,\delta_P)}$ (whose class is γ_{P,δ_P}) is the restriction of a projective representation $\tilde{\delta}_P^e$ of $\mathcal{W}^e_{(P,\delta_P),(P,\delta_P)}$ (which is defined as usual, by twisting $\delta_P = \delta_P^{A,e} \otimes \delta_P^{B,e}$ with the automorphisms of $\mathcal{H}_P$ coming from $\mathcal{W}^e_{(P,\delta_P),(P,\delta_P)}$). Observe that $\tilde{\delta}_P^e$ is the tensor product of a projective representation $\tilde{\delta}_P^{A,e}$ of $\mathcal{W}^{A,e}_{(P,\delta_P),(P,\delta_P)}$ and a projective representation $\tilde{\delta}_P^{B,e}$ of $\mathcal{W}^{B,e}_{(P,\delta_P),(P,\delta_P)}$. The first tensor factor is linear because it has dimension 1 (and thus is trivial as a projective representation), and the second tensor factor is linear since $H^2(C_2,\mathbb{C}^\times) = 1$ (actually $H^2(C_n,\mathbb{C}^\times) = 1$ for any finite cyclic group C_n, see e.g. [15, Exercise XX 16] (Warning: the even and odd cases have been mixed up in loc.cit.)). Hence by restriction we see that $[\gamma_{P,\delta_P}] = 1$, which is what we needed to show. □

6.4. R_0 of type C_n

Proposition 6.7. *Suppose that R_0 is of type C_n. If X is the weight lattice of R_0 we denote the corresponding affine Hecke algebra $\mathcal{H}_n^{C,e}$. In this case we have $[\gamma_{W,\Delta}] = 1$.*

Proof. In this case $\mathcal{H}_n^{C,e}$ is simply a specialization of the three parameter type $\mathrm{C}_n^{(1)}$ affine Hecke algebra, hence the result follows from Proposition 6.5. □

Proposition 6.8. *Suppose that $\mathcal{H}$ is the non-extended affine Hecke algebra $\mathcal{H}_n^{C,Q}$ i.e. R_0 is of type C_n and the lattice X equals the root lattice of R_0. Then $[\gamma_{\mathcal{W},\Delta}] = 1$.*

Proof. Again we compare the situation with the standard case $\mathrm{C}_n^{(1)}$. The description of the standard parabolic subsystems $P \subset F_0$ is as before, where the rightmost factor of type B_l has to be replaced by C_l of course. The new complication is that the affine Hecke algebra $\mathcal{H}_P$ is not always a tensor product of extended type A-factors and (possibly) a type C factor. If at least one of the λ_i is odd then $\mathcal{H}_P$ is as before, a tensor product of a number of extended type A factors $\mathcal{H}^A_{\lambda_{i-1}}(\Lambda_{\lambda_{i-1}}, q_1)$ with at most one type C factor $\mathcal{H}_l^{C,e}$. In other words, the lattice associated with the Hecke algebra $\mathcal{H}_P$ is the weight lattice Λ_P of R_P. However if the λ_i are all even, the algebra $\mathcal{H}_P$ is an index two subalgebra of the affine Hecke algebra just described obtained by taking the fixed points with respect to the twisting involution ψ_ϵ corresponding to the unique W_P-invariant quadratic character ϵ of the weight lattice Λ_P of R_P which is

nontrivial when restricted to any of the weight lattices of the irreducible direct summands of R_P.

In the first case (if there exists at least one odd λ_i) then the group K_P^{C,Λ_r} is of the form $K_P \times C_2$, where the last factor C_2 acts on the extended algebra $\mathcal{H}_l^{C,e}$ via the twisting automorphism associated with the nontrivial character of the weight lattice Λ^C of type C_l trivial on the root lattice Λ_r^C. The argument to see the triviality of $\gamma_{\mathcal{W},\Delta}$ is now exactly analogous to the proof of Proposition 6.6.

In the second case (when all λ_i are even) then K_P^{C,Λ_r} is the quotient of the previously described group by the subgroup $K_\epsilon = \langle \epsilon \rangle$ generated by ϵ. The second case can be reduced to the first case as follows. Let $\mathcal{H}_P^e$ be the semisimple affine Hecke algebra whose root system is R_P and whose lattice X_P^e equals the weight lattice Λ_P of R_P. Then K_ϵ acts on $\mathcal{H}_P^e$ and $\mathcal{H}_P = (\mathcal{H}_P^e)^{K_\epsilon}$. We may assume that $P \neq F_0$, otherwise there is nothing to prove. But then, by definition, X_P contains factors of the form $\Lambda^A_{\lambda_i - 1}$ with $\lambda_i \geq 2$. Thus ϵ is a W_P-invariant character of X_P^e, and is nontrivial on each of the type A factors. Now we use the following special feature of the affine Hecke algebras $\mathcal{H}^{A,X^A}_{\lambda_i-1}$ of type A_{λ_i-1} (and any lattice $X^A_{\lambda_i-1}$): For such affine Hecke algebras, twisting by a nontrivial $W^A_{\lambda_i-1,0}$-invariant character $k \in T^A_{\lambda_i-1}$ has no fixed points on the set of equivalence classes of discrete series characters $\Delta^A_{\lambda_i-1}$ (this follows from considering the unitary part of the central characters of the discrete series characters, as in the proof of Proposition 6.5). Since $\mathcal{H}_P^e$ is merely a product of such type A-factors and a type C factor, this shows in view of the above that "twisting by ϵ" acts on the set Δ_P^e of equivalence classes of discrete series representations of $\mathcal{H}_P^e$ *without fixed points.* In turn this implies (by elementary Clifford theory, see [27], and Lemma 6.3) that the restriction functor sends irreducible discrete series of $\mathcal{H}_P^e$ to irreducible discrete series of $\mathcal{H}_P$, and all discrete series of $\mathcal{H}_P$ are obtained in this way. Hence the action groupoid $\mathcal{W}^e_{P,\Delta_P^e}$ of the group $K_P^e \rtimes \mathfrak{W}_{P,P}$ acting on the set of equivalence classes Δ_P^e of irreducible discrete series representations of $\mathcal{H}_P^e$ via twisting automorphisms on $\mathcal{H}_P^e$, is Morita equivalent to $\mathcal{W}_{P,\Delta_P}$. But $\mathcal{W}^e_{P,\Delta_P^e}$ is a union of groups $\mathcal{W}^e_{(P,\tilde{\delta}_P),(P,\tilde{\delta}_P)}$ which are of the same form as in the first case, reducing the second case to the first case as required. □

6.5. R_0 of type $\mathbf{D}_n$ and $X \neq \Lambda_r$

Proposition 6.9. *Suppose that $\mathcal{H}$ is an affine Hecke algebra of type D_n whose lattice X is not the root lattice. Then $[\gamma_{\mathcal{W},\Delta}] = 1$.*

Proof. Again the standard parabolic subsystems R_P of R_0 are products of a number of type A factors and at most one type D factor. The complicating aspect in the present case is the more complicated structure of the group of automorphisms of the type D factor that needs to be considered.

First let us suppose that the lattice $X = \Lambda$, the weight lattice of R_0. In this case the description of the group $K_P^{D,\Lambda}$ similar to $K_P^{B,\Lambda}$ as in the proof of Proposition 6.6, where we may and will assume now that $4 \leq l < n$, since otherwise R_P either equals R_0 (and there is nothing to prove in this case) or has only type A-factors (which brings us to the situation of Proposition 6.4). We write

$$\Lambda = \Lambda_n^D = \Lambda_0 + \Lambda_1 = \mathbb{Z}^n + ((1/2, 1/2, \ldots, 1/2) + \mathbb{Z}^n) \tag{6.5}$$

If there exists at least one odd λ_i then $K_P^{D,\Lambda}$ is a direct product of the cyclic groups corresponding to the type A-factors (hence the corresponding twisting automorphisms act trivially on the type D-factor) and a factor $C_2 = \langle\eta\rangle$ acting nontrivially only on the type D-factor, where $\eta \in K_l^D(\Lambda_l^D)$ is the unique character of Λ_l^D with kernel $\mathbb{Z}^l$ (acting by twisting on the tensor factor $\mathcal{H}_l^D(\Lambda)$ of $\mathcal{H}_P$). If all λ_i are even, all the direct factors of the group just described are even cyclic groups, and thus carry a unique nontrivial quadratic character. Then $K_P^{D,\Lambda}$ is the kernel of the product ρ of all these nontrivial characters on the cyclic factors.

Let M be the largest part of $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n - l$, and let μ_i (for $i = 1, \dots, M$) be the multiplicity of i as a part of λ. Then it is easy to see that

$$\mathfrak{W}_{P,P} = (W_0(\mathrm{B}_{\mu_1}) \times \cdots \times W_0(\mathrm{B}_{\mu_M}) \times \langle\omega\rangle)^{\Sigma_{odd}} \tag{6.6}$$

where ω is the restriction of the unique nontrivial diagram automorphism of D_n to the subdiagram of type D_l, and where Σ_{odd} is the product over all factors $W_0(\mathrm{B}_{\mu_{2j+1}})$ of the linear character Σ_{2j+1} given by taking the product of the signs of a signed permutation and, in the last factor, of the unique nontrivial character of $\langle\omega\rangle \approx C_2$.

In all cases we define an extension $\mathcal{W}_{P,P}^e \supset \mathcal{W}_{P,P}$ of order 2, where $\mathcal{W}_{P,P}^e$ is of the form

$$\mathcal{W}_{P,P}^e = \mathcal{W}_{P,P}^{A,e} \times \mathcal{W}_{P,P}^{D,e} \tag{6.7}$$

with

$$\mathcal{W}_{P,P}^{D,e} = \langle\omega\rangle \times \langle\eta\rangle \approx C_2 \times C_2 \tag{6.8}$$

if λ contains odd parts, and

$$\mathcal{W}_{P,P}^{D,e} = \langle\eta\rangle \approx C_2 \tag{6.9}$$

if λ has only even parts. In the latter case it is again clear that $[\gamma_{(P,\delta_P)}] = 1$ as in Proposition 6.6. In the first case we need to show that if a discrete series representation δ_P^D of $\mathcal{H}_l^D(\Lambda_l^D)$ contains $\langle\omega\rangle\times\langle\eta\rangle$ in its isotropy group then it can be extended to a representation of $\mathcal{H}_l^D(\Lambda_l^D)\rtimes(\langle\omega\rangle \times \langle\eta\rangle)$. This follows from Lemma 6.10.

Next we assume that $X = \mathbb{Z}^n$. With the previous situation $X = \Lambda$ in mind this case is easier, since everything is the same except that K_P does not involve the factor $\langle\eta\rangle$ now (compare with the proof of Proposition 6.5). The above arguments apply in this simpler situation as well (but the extension of δ_P^D is obvious now, since its isotropy is at most a C_2) showing that $[\gamma] = 1$ in this case.

Finally if n is even, we need to consider two more lattices $X = \Lambda_-$ and Λ_+ with

$$\Lambda_\pm = \Lambda_r + ((1/2, 1/2, \dots, \pm 1/2) + \Lambda_r) \tag{6.10}$$

Let $\Lambda_P = \Lambda_\lambda^A \times \Lambda_l^D$ be the weight lattice of R_P. Observe that the second projection of $X_P \subset \Lambda_P$ is equal to the full weight lattice Λ_l^D of type D_l (unless $l = n$, a case which we excluded at the start of this proof). Let K_P^Λ be the subgroup of the group of characters of the weight lattice Λ_P which restrict to 1 on the sublattice $X \cap \mathbb{R}P$. Restriction of characters of Λ_P to X_P induces a quotient map $q : K_P^\Lambda \mapsto K_P$. The above observation implies that the first projection $p_1 : K_P^\Lambda \to K_\lambda^A$ (by restriction of a characters of Λ_P to the factor Λ_λ^A) is injective on the kernel of q. Hence, the arguments in the proof of Proposition 6.8 (reducing the "second case" to the "first case") apply and show that we can replace the quotient K_P by K_P^Λ via a suitable equivalence of groupoids. In this situation we may write $\delta_P^e = \delta_P^{A,e} \otimes \delta_P^{D,e}$ for the extension to $\mathcal{H}_P^e$ of the discrete series representation δ_P of $\mathcal{H}_P$. The group $\mathcal{W}_{P,P}^e = K_P^e \rtimes \mathfrak{W}_{P,P}^e$

of automorphisms of $\mathcal{H}_P^e$ that needs to be considered now is always a subgroup (depending on P) of the automorphism group $\mathcal{W}_{P,P}^m = \mathcal{W}_{P,P}^{A,m} \times \mathcal{W}_{P,P}^{D,m}$ of $\mathcal{H}_P^e$ described by

$$\mathcal{W}_{P,P}^{A,m} = K_P^{A,m} \rtimes \mathfrak{W}_{P,P}^A \tag{6.11}$$

with

$$\mathfrak{W}_{P,P}^A = W_0(\mathrm{B}_{\mu_1}) \times \cdots \times W_0(\mathrm{B}_{\mu_M}); K_P^{A,m} = C_2^{\mu_2} \times \cdots \times C_M^{\mu_M}, \tag{6.12}$$

and with

$$\mathcal{W}_{P,P}^{D,m} = K_P^{D,m} \rtimes \langle \omega \rangle \tag{6.13}$$

where $K_P^{D,m}$ is the character group of $\Lambda_l^D/\Lambda_{r,l}^D$ (a group of order 4). Therefore, arguing as in the proof of Proposition 6.6, it suffices to show that $\delta_P^{D,e}$ extends to a linear representation of $\mathcal{H}_P^{D,e} \rtimes I_{(P,\delta_P^{D,e})}$, where $I_{(P,\delta_P^{D,e})}$ is the isotropy group of $[\delta_P^{D,e}]$ in $\mathcal{W}_{P,P}^{D,m}$. This follows from Lemma 6.10, finishing the proof. $\square$

The following lemma uses a nontrivial property of irreducible discrete series representations of the graded affine Hecke algebra type D_n proved in [26].

Lemma 6.10. *Let $\mathcal{H}$ be the affine Hecke algebra of type D_l (with $l \geq 4$) which is maximally extended, i.e. $X = \Lambda_l^D$, the weight lattice of R_0. For convenience we take the standard realization of R_0, with basis $\{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$. Let K be the group of characters of $\Lambda_l^D/\Lambda_{r,l}^D$ (a group of order 4) and let ω be the diagram automorphism of R_0 induced by the orthogonal reflection in the hyperplane $x_n = 0$. We let the group $\mathcal{W} := K \rtimes \langle \omega \rangle$ act on $\mathcal{H}$ by twisting automorphisms as usual. Let $\delta \in \Delta(\mathcal{H})$, and let I_δ be the isotropy group of $[\delta]$ in $\mathcal{W}$. Then δ extends to a representation of $\mathcal{H} \rtimes I_\delta$.*

Proof. We first observe that $\mathcal{W} \approx \mathbb{D}_8$, the dihedral group of order 8. Let $\epsilon \in \mathcal{W}$ be an element of order 4, and let $\eta = \epsilon^2$ (the generator of the center of $\mathcal{W}$). Then $\mathcal{W} = \langle \epsilon, \omega \rangle$. We define $\kappa := \omega\epsilon$, an element of order 2. There are 3 subgroups of index 2 in $\mathbb{D}_8$, one of which is cyclic. If l is odd then $K = \langle \epsilon \rangle \approx C_4$; if l is even then $K = \langle \eta, \kappa \rangle \approx C_2 \times C_2$. The other subgroup of index 2 is $N = \langle \eta, \omega \rangle \approx C_2 \times C_2$.

It follows from [26, Theorem 7.1, Theorem 8.7] that any $\delta \in \Delta(\mathcal{H})$ admits (precisely two) extensions (δ_-, δ_+ say) to $\mathcal{H} \rtimes \langle \omega \rangle = \mathcal{H}_l^B(\Lambda_l^B)$ (with $q_{2x_n} = 1$). In particular, we always have $\omega \in I_\delta$. It follows that either I_δ is cyclic (in which case the desired result is obvious, since cyclic groups have a trivial Schur multiplier (see the remark at the end of the proof of Theorem 6.5)) or $N \subset I_\delta$.

We now use [12] that the Schur multiplier of the group $\mathbb{D}_8$ is C_2, and that [28, Section 3] the restriction of its unique nontrivial class $[\alpha] \in H^2(\mathbb{D}_8, \mathbb{C}^\times)$ to both subgroups of type $C_2 \times C_2$ is nontrivial. Combined with the above remarks we see that it suffices to prove that δ extends to $\mathcal{H} \rtimes N$ if $N \subset I_\delta$, which is what we will assume from now on.

As we have already remarked, δ extends to irreducible discrete series representations $\delta_\pm$ of $\mathcal{H}_l^B(\Lambda_l^B)$ (with $q_{2x_n} = 1$). This algebra admits an involutive automorphism η^B whose fixed point set is $\mathcal{H}_l^B(\mathbb{Z}^l)$ (with $q_0 = q_2 = 1$; recall that $\Lambda_{r,l}^B = \mathbb{Z}^l$) which fixes ω and which restricts to η on $\mathcal{H}_l^D(\Lambda_l^D) \subset \mathcal{H}_l^B(\Lambda_l^D)$. To prove that δ extends to $\mathcal{H} \rtimes N$ it suffices to show that η^B is in the isotropy group of $[\delta_\pm]$. We claim actually that this holds true for any $\delta \in \Delta(\mathcal{H}_l^B(\Lambda_l^B))$. Indeed, applying [26, Theorem 7.1, Theorem 8.7] to $\mathcal{H}_l^B(\Lambda_l^B)$ (with $q_{2x_n} = 1$) and to its fixed point algebra $\mathcal{H}_l^B(\mathbb{Z}^l) = \mathcal{H}(\mathrm{C}_l^{(1)})$ (with $q_0 = q_2$) (see [26, Example 8.3]) under η^B we see that

$|\Delta_l^B(\mathbb{Z}^l)| = 2|\Delta_l^B(\Lambda_l^B)|$. In view of Lemma 6.3 and [27, Theorem A.13] we conclude on the other hand that if there would exist a $\delta \in \mathcal{H}_l^B(\Lambda_l^B)$ whose class is not invariant for η^B then we would necessarily have $|\Delta_l^B(\mathbb{Z}^l)| < 2|\Delta_l^B(\Lambda_l^B)|$, proving our claim and thus finishing the proof of the Lemma. □

6.6. R_0 of type D_n and $X = \Lambda_r$

In the one remaining classical case, the affine Hecke algebra of type D_n with $X = \Lambda_{r,n}^D$, the root lattice of R_0, it is (remarkably) not always true that that $[\gamma_{\mathcal{W},\Delta}] = 1$. Yet we have:

Proposition 6.11. *Let $\mathcal{H} = \mathcal{H}_n^D(\Lambda_{r,n})$ and let $\mathcal{W}_{\Xi_u}$ be its groupoid of unitary standard induction data. Then $[\gamma_{\mathcal{W},\Xi}] = 1$.*

Proof. Let $\xi = (P, \delta, t) \in \Xi_u$. We need to show that the 2-cocycle γ_ξ of $\mathcal{W}_{\xi,\xi}$ is a coboundary. Let P be as in (6.2) with $4 \leq l < n$ and as before, let μ_i denote the multiplicity of the part i in $\lambda \vdash n - l$. The group $\mathfrak{W}_{P,P}$ does not depend on the lattice X, so is still given by (6.6). Let us write (6.6) as

$$\mathfrak{W}_{P,P} = (\mathfrak{W}_{P,P}^A \times \mathfrak{W}_{P,P}^D)^{\Sigma_{odd}} \tag{6.14}$$

with $\mathfrak{W}_{P,P}^D = \langle \omega \rangle$ (ω being the unique nontrivial automorphism of the diagram of type D_l that extends to D_n) and

$$\mathfrak{W}_{P,P}^A = W_0(\mathrm{B}_{\mu_1}) \times \cdots \times W_0(\mathrm{B}_{\mu_M}) \tag{6.15}$$

and Σ_{odd} the linear character defined in the text just below (6.6). Recall that Σ_{odd} is trivial on $\mathfrak{W}_{P,P}^A$ iff all parts of λ are even. We introduce the projections

$$\pi_{\mathfrak{W}}^A : \mathfrak{W}_{P,P} \to \mathfrak{W}_{P,P}^A, \quad \pi_{\mathfrak{W}}^D : \mathfrak{W}_{P,P} \to \mathfrak{W}_{P,P}^D \tag{6.16}$$

Notice that $\pi_{\mathfrak{W}}^A$ is an isomorphism (always) and $\pi_{\mathfrak{W}}^D$ is trivial iff all parts of λ are even. We have by definition

$$\mathcal{W}_{P,P} = K_P \rtimes \mathfrak{W}_{P,P} \tag{6.17}$$

Let us now compute the lattice X_P and the abelian group K_P (this is similar to the proof of Proposition 6.8). The orthogonal projection X_P of the root lattice $\Lambda_{r,n}^D$ onto $\mathbb{R}R_P$ is the product of the weight lattice of the type A factors of R_P with the lattice $\mathbb{Z}^l$ for the type D_l factor of R_P, provided that λ has odd parts. Hence in this case we have

$$\mathcal{H}_P = \mathcal{H}_P^A(\Lambda_P) \otimes \mathcal{H}_l^D(\mathbb{Z}^l) \tag{6.18}$$

If λ has only even parts then X_P is a sublattice of index two of the lattice just described, namely the kernel of the product ϵ of the unique nontrivial W_P-invariant quadratic character of the direct summands of the above lattice corresponding to the irreducible components of R_P. Hence if λ has odd parts then

$$K_P = K_P^A \times K_P^D \tag{6.19}$$

where

$$K_P^A \approx C_2^{\mu_2} \times \cdots \times C_M^{\mu_M} \tag{6.20}$$

and where

$$K_P^D := \langle \kappa \rangle \tag{6.21}$$

with κ the unique nontrivial character of the lattice $\mathbb{Z}^l$ which is trivial on the root lattice $\Lambda^D_{r,l}$ of the root system D_l. In this case we denote by

$$\pi^A_K : K_P \to K^A_P, \quad \pi^D_K : K_P \to K^D_P \tag{6.22}$$

the projections onto the type A and the type D factors. We remark that π^D_K is invariant for the action of $\mathcal{W}_{P,P}$ on K_P. If λ is even then K_P is the quotient of this group by $\langle\epsilon\rangle$. If λ has odd parts then we define

$$\pi^D = \pi^D_{\mathfrak{W}} \times \pi^D_K : \mathcal{W}_{P,P} \to \mathcal{W}^D_{P,P} := \mathfrak{W}^D_{P,P} \times K^D_P \subset \mathrm{Aut}(\mathcal{H}^D_l(\mathbb{Z}^l)) \tag{6.23}$$

Observe that $\mathcal{W}^D_{P,P} = \langle\omega\rangle \times \langle\kappa\rangle \approx C_2 \times C_2$. Similarly we define

$$\pi^A = \pi^A_{\mathfrak{W}} \times \pi^A_K : \mathcal{W}_{P,P} \to \mathcal{W}^A_{P,P} := \mathfrak{W}^A_{P,P} \ltimes K^A_P \subset \mathrm{Aut}(\mathcal{H}^A_P(\Lambda_P)) \tag{6.24}$$

If λ is even, then (as in the proof of Proposition 6.8) we extend the lattice X_P of $\mathcal{H}_P$ to obtain $\mathcal{H}^e_P \supset \mathcal{H}_P$ (a quadratic extension) whose associated lattice X^e_P is the product of the weight lattices of the type A factors of R_P with the lattice $\mathbb{Z}^l$ for the type D_l factor of R_P. As in Proposition 6.8 this leads to a groupoid $\mathcal{W}^e_{P,\Delta^e_P}$ which is Morita equivalent to $\mathcal{W}_{P,\Delta}$. Using that $\pi^D_{\mathfrak{W}} = 1$ if λ is even we can now apply the same argument as given in the proof of Proposition 6.8 to conclude that γ^e_{P,Δ^e} is trivial. As we know this implies the triviality of $\gamma_{\mathcal{W}_{P,P},\Xi_P}$ as well in this case.

So let us assume from now that λ contains at least one odd part. To prove the triviality of $\gamma_{\mathcal{W}_{P,P},\Xi_P}$ it is enough to prove the triviality of γ_ξ for the isotopy group $\mathcal{W}_{\xi,\xi}$ of an arbitrary object $\xi = (P, \delta_P, t)$ with $\delta_P = \delta^{A_{\lambda_1-1}} \otimes \cdots \otimes \delta^{A_{\lambda_r-1}} \otimes \delta$. Let us therefore consider the action of $\mathcal{W}_{P,P}$ on the space Ξ_P of parameters first. Using the action of K^A_P we may and will assume that the one dimensional type A discrete series representations $\delta^{A_{\lambda_i-1}}$ all have a real central character.

In the present situation we have $T = (\mathbb{C}^\times)^n/\langle\pm 1\rangle$. Recall that $T^P \subset T$ is the subtorus of the characters of the orthogonal projection of $X = \Lambda^D_{r,n}$ onto the subspace $\mathbb{R}P^\perp$. For each part i of λ this projection is generated by generators $E^i_1, \dots, E^i_{\mu_i}$ say, which we normalize by requiring that E^i_j has coordinates $1/i$ at the i slots corresponding to the j-th part of size i of λ, while its remaining coordinates are 0. Accordingly, for the element in $t \in T^P$ such that $t(E^i_j) = t^i_j \in \mathbb{C}^\times$ we write:

$$t = (t^1_1, t^1_2, \dots, t^1_{\mu_1}, t^2_1, t^2_2, \dots, t^{M-1}_{\mu_{M-1}}, t^M_1, t^M_2, \dots, t^M_{\mu_M}) \tag{6.25}$$

In order to see how K_P acts on T^P and on Δ_P we need to identify $K_P = T_P \cap T^P$ as a subgroup of T^P explicitly. The group K^A_P is the subgroup of elements t with $(t^i_j)^i = 1$, and acts by multiplication on T^P. However, there exists an additional generator $\kappa \in T^P \cap T_P \subset T$ (since λ is not even), which has its first $n-l$ coordinates (in $T = (\mathbb{C}^\times)^n/\langle\pm 1\rangle$) equal to 1 and its last l coordinates equal to -1. Indeed, this description makes it obvious that $\kappa \in T_P$. On the other hand, κ can also be given by a row of $n-l$ coordinates equal to -1 and a tail of l coordinates equal to 1. This description makes it obvious that $\kappa \in T^P$. Hence we have $\kappa \in T_P \cap T^P$. By (6.19) K_P is generated by K^A_P and κ. In the coordinates (6.25) on T^P the element $\kappa \in T_P \cap T^P$ equals $-1 \in T^P$ (i.e. $\kappa^i_j = -1$ for all i, j). The subgroup $K_P \subset T^P$ is thus given by those $t \in T^P$ with either $(t^i_j)^i = 1$ (for all i), or with $(t^i_j)^i = -1$ (for all i). The projection $\pi^D_K : K^P \to K^D_P$ is given by $\pi^D_K(k) = \kappa$ if $(k^i_j)^i = -1$ for some pairs (i,j)

(hence all) with i odd, and $\pi_K^D(k)=1$ else. The projection onto K_P^A is given by $\pi_K^A(k)=k$ if $\pi^D(k)=1$, and $\pi_K^A(k)=-k$ else.

The group $\mathfrak{W}_{P,P}$ acts on T^P by signed permutations on the E_j^i which leave the superscript i unchanged. If $g\in\mathcal{W}_{P,P}$ and $\xi=(P,\delta,t)$ then $g\xi:=(P,\delta^g,gt)$, where $\delta^g\simeq\delta\circ\phi_g$. By (6.18), $\delta=\delta^A\otimes\delta^D$ and thus

$$\delta^g=(\delta^A\otimes\delta^D)^g=(\delta^A)^{\pi^A(g)}\otimes(\delta^D)^{\pi^D(g)} \tag{6.26}$$

Let us show first that $[\gamma_\xi]=1$ if the map $\pi^D|_{\mathcal{W}_{\xi,\xi}}\to\mathcal{W}_{P,P}^D$ is not surjective. We extend $\mathfrak{W}_{P,P}$ to $\mathfrak{W}_{P,P}^e=\mathfrak{W}_{P,P}^A\times\mathfrak{W}_{P,P}^D$ and accordingly define $\mathcal{W}_{P,P}^e=\mathcal{W}_{P,P}^A\times\mathcal{W}_{P,P}^D$. By the assumption we see that $\mathcal{W}_{\xi,\xi}\subset\widetilde{\mathcal{W}}_{P,P}:=\mathcal{W}_{P,P}^A\times C_2$ where $C_2\subset\mathcal{W}_{P,P}^D$ is a suitably chosen subgroup. But then the projective representation of $\widetilde{\mathcal{W}}_{P,P}$ defines a trivial 2-cocycle (as in the proof of Proposition 6.6, using also (6.18)), hence in particular $[\gamma_\xi]=1$ in this case.

So let us now assume that $\pi^D|_{\mathcal{W}_{\xi,\xi}}$ is onto $\mathcal{W}_{P,P}^D$. By (6.18) this implies in particular that $(\delta^D)^\kappa=\delta^D$ and $(\delta^D)^\omega=\delta^D$. Thus δ^D defines a 2-cocycle γ_δ of $\mathcal{W}_{P,P}^D$. By (6.26) the desired result follows from the claim: *The pullback of γ_δ under the restriction $\pi_{\xi,\xi}^D:\mathcal{W}_{\xi,\xi}\to\mathcal{W}_{P,P}^D$ of the map π^D of (6.23) is a coboundary.* The remaining part of this proof is devoted to the proof of this claim.

The condition for $g=kw\in K_P\rtimes\mathfrak{W}_{P,P}$ to be in $\mathcal{W}_{\xi,\xi}$ is that $(\delta^A)^{\pi^A(g)}=\delta^A$ and $gt=t$. Since $\mathfrak{W}_{P,P}^A$ fixes δ^A (by our choice to take the central character of δ^A real) and K_P^A acts freely on Δ_P^A, the first equation is equivalent to $\pi^A(k)=1$. Hence $k=\pm1\in T^P$ (recall that $-1\in T^P$ is the element $\kappa\in K_P$), and accordingly $\pm wt=t$. Hence we have

$$\mathcal{W}_{\xi,\xi}\approx\{w\in\mathfrak{W}_{P,P}^A=W_0(B_{\mu_1})\times\cdots\times W_0(B_{\mu_1})\mid wt=\pm t\} \tag{6.27}$$

and in this realization the homomorphism $\pi_{\xi,\xi}^D:\mathcal{W}_{\xi,\xi}\to\mathcal{W}_{P,P}^D$ is given by $\pi_{\xi,\xi}^D(w)=\omega^{\sigma(w)}\times\kappa^{\epsilon(w)}$ where $\sigma,\epsilon:\mathcal{W}_{\xi,\xi}\to C_2$ are two linear characters defined by $(-1)^{\sigma(w)}=\Sigma_{odd}(w)$ (with Σ_{odd} as in (the proof of) Proposition 6.9), and $wt=(-1)^{\epsilon(w)}t$.

The torus T^P is a direct product $T^P=\prod_{i=1}^r T^{P,(i)}$ of the tori $T^{P,(i)}$ of characters of the root lattice of B_{μ_i}. Now $T^{P,(i)}$ has a double cover $\tilde{T}^{P,(i)}$, the torus of characters of the weight lattice of B_{μ_i}. The kernel of the covering map is denoted by $\langle\eta^{(i)}\rangle$ where $\eta^{(i)}$ is the unique nontrivial $W_0(\mathrm{B}_{\mu_i})$-invariant element of $\tilde{T}^{P,(i)}$. Thus we have

$$1\to\langle\eta^{(i)}\rangle\to\tilde{T}^{P,(i)}\to T^{P,(i)}\to1 \tag{6.28}$$

Putting these together we get an exact sequence

$$1\to\langle\eta^{(1)},\dots,\eta^{(r)}\rangle\to\tilde{T}^P\to T^P\to1 \tag{6.29}$$

where $\langle\eta^{(1)},\dots,\eta^{(r)}\rangle\approx C_2^r$. By (6.27) we see that the action of $\mathcal{W}_{\xi,\xi}$ on T^P extends to $\tilde{T}^P$.

Consider the set $S_t=\{s\in\tilde{T}^P\mid s\to\{\pm t\}\subset T^P\}$. This set admits a free, transitive action (by multiplication) of the subgroup $M:=\langle\eta^{(1)},\dots,\eta^{(r)}\rangle\times\langle\tilde{\kappa}\rangle$ of $\tilde{T}^P$, where $\tilde{\kappa}$ denotes a lift of $-1=\kappa\in T^P$. This abelian group is isomorphic to $C_2^{r-1}\times C_4$ if λ contains parts with an odd multiplicity and is isomorphic to C_2^{r+1} otherwise. Clearly M is stable for the action of $\mathcal{W}_{\xi,\xi}$ on $\tilde{T}^P$, making M a module over $\mathcal{W}_{\xi,\xi}$. To describe the module structure explicitly, remark that the element $\tilde{\kappa}$ is not $W_0(B_{\mu_1})\times\cdots\times W_0(B_{\mu_1})$-invariant. In fact, if $w=w^{(1)}\times\cdots\times w^{(r)}\in W_0(B_{\mu_1})\times\cdots\times W_0(B_{\mu_1})$ then

$$w(\tilde{\kappa})=\eta(w)\tilde{\kappa} \tag{6.30}$$

where

$$\eta(w) = \prod_{i=1}^{r} (\eta^{(i)})^{\sigma_i} \in \langle \eta^{(1)}, \ldots, \eta^{(r)} \rangle \tag{6.31}$$

with $(-1)^{\sigma_i} = \Sigma_i(w^{(i)})$, where Σ_i is the character on $W_0(\mathrm{B}_{\mu_i})$ whose kernel is $W_0(\mathrm{D}_{\mu_i})$. The elements $\eta^{(i)}$ are all fixed for the action of $\mathcal{W}_{\xi,\xi}$.

The M-orbit S_t is stable for the action of $\mathcal{W}_{\xi,\xi}$ on $\tilde{T}^P$ as well. Hence, any lift $\tilde{t} \in S_t$ of t defines a 1-cocycle $\mu : \mathcal{W}_{\xi,\xi} \to M$ of $\mathcal{W}_{\xi,\xi}$ with values in M by the formula $w\tilde{t} = \mu(w)\tilde{t}$. We fix such a lift $\tilde{t}$ once and for all. Consider the abelian group $N := \langle \eta, \tilde{\kappa} \rangle$ defined by the relations $\eta^2 = 1$ and $\tilde{\kappa}^2 = \eta$ (if the number of odd i with μ_i odd is odd) or else $\tilde{\kappa}^2 = 1$ (hence N is isomorphic to C_4 if there are an odd number of odd i with odd μ_i, and N is isomorphic to $C_2 \times C_2$ else). Consider the $\mathcal{W}_{\xi,\xi}$-module structure on N defined by $w(\eta) = \eta$ for all $w \in \mathcal{W}_{\xi,\xi}$, and by

$$w(\tilde{\kappa}) = \eta^{\sigma(w)} \tilde{\kappa} \tag{6.32}$$

(the fact that this defines a $\mathcal{W}_{\xi,\xi}$-module is equivalent to saying that σ is a character of $\mathcal{W}_{\xi,\xi}$). The module N is a quotient of M via the unique homomorphism $\alpha : M \to N$ satisfying $\alpha(\tilde{\kappa}) = \tilde{\kappa}$, $\alpha(\eta^{(i)}) = \eta$ if i is odd, and $\alpha(\eta^{(i)}) = 1$ if i is even (in fact, the definition of N is such that α exists). The cocycle μ induces a cocycle of $\mathcal{W}_{\xi,\xi}$ with values in N which we denote by μ_N. Consider the following diagram

$$\begin{tikzcd}
 & N \arrow[d, hook, "i"'] & \mathcal{W}_{\xi,\xi} \arrow[dl, dotted, "\chi"'] \arrow[d, "\pi^D_{\xi,\xi}"] & \\
\langle \eta \rangle \arrow[r, hook] & \mathbb{D}_8 \arrow[r, two heads, "j"'] \arrow[d, shift right] & \mathcal{W}^D_{P,P} \arrow[r, equal] & \langle \omega \rangle \times \langle \kappa \rangle \\
 & \langle \omega \rangle \arrow[u, shift right] & &
\end{tikzcd} \tag{6.33}$$

where $\mathbb{D}_8 = \langle \tilde{\omega}, \tilde{\kappa} \rangle$ is a dihedral group of order 8 in which $\tilde{\omega}$ has order two and η (or more precisely $i(\eta)$) is the nontrivial central element. The defining relations in $\mathbb{D}_8$ are given by $\tilde{\omega}\tilde{\kappa}\tilde{\omega} = \eta\tilde{\kappa}$. The map j is defined by requesting that $j(\tilde{\omega}) = \omega$ and $j(\tilde{\kappa}) = \kappa$.

We claim that there exists a homomorphism $\chi : \mathcal{W}_{\xi,\xi} \to \mathbb{D}_8$ as indicated in the diagram. Obviously the vertical exact sequence is split. We choose the splitting $\omega \to \tilde{\omega}$ of this sequence. The homomorphism $\pi^D_{\xi,\xi}$ gives rise to a homomorphism $\pi : \mathcal{W}_{\xi,\xi} \to \mathbb{D}_8$ with image $\langle \tilde{\omega} \rangle$ obtained by composing $\pi^D_{\xi,\xi}$ with the first projection $\langle \omega, \kappa \rangle \to \langle \omega \rangle$ and the lift $\omega \to \tilde{\omega}$. We can write this explicitly by $\pi(w) = \tilde{\omega}^{\sigma(w)}$. We now define a map $\chi : \mathcal{W}_{\xi,\xi} \to \mathbb{D}_8$ by

$$\chi(w) = i(\mu_N(w))\pi(w). \tag{6.34}$$

We claim that χ is a group homomorphism. Indeed, the action of $\mathcal{W}_{\xi,\xi}$ on N is related to π by the formula $i(n^w) = \pi(w)i(n)\pi(w)^{-1}$ (using the explicit formulas for the module N and for π). It follows that (6.34) indeed defines a homomorphism. Next we claim that χ makes (6.33) commutative as indicated. Indeed, $i(\mu_N(w)) \equiv \tilde{\kappa}^{\epsilon(w)} \bmod \langle \eta \rangle$ as follows from the definition of μ (and μ_N) and of ϵ. On the other hand, by construction of π we see that $\pi(w) \equiv \tilde{\omega}^{\sigma(w)} \bmod \langle \eta \rangle$. Together these two congruences (modulo the center $\langle \eta \rangle$ of $\mathbb{D}_8$) imply the claim.

Since $\mathbb{D}_8$ is the Schur extension of $\langle\omega,\kappa\rangle$ it now follows that the pullback of γ_δ under the homomorphism $\pi^D_{\xi,\xi}$ is indeed a coboundary (since $\pi^D_{\xi,\xi}$ factors through the Schur extension map), finishing the proof of the claim and of the Theorem. $\square$

6.7. **Final remarks**

6.7.1. *Multiplicity one W_0-types.* We would like to comment on a natural alternative approach to proving the triviality of the cocycles γ_ξ for $\xi = (P,\delta,t) \in \Xi_u$. The equivalence class of the restriction to $\mathcal{H}(W_0,q_0)$ of $\pi(\xi)$ is independent of the continuous parameter $t \in T^P$. We will refer to an irreducible representation of $\mathcal{H}(W_0,q_0)$ as a "W_0-type" in this paragraph. If there exists a W_0-type appearing in $(V_\xi,\pi(\xi))$ with multiplicity one then we can normalize the action of $\mathfrak{R}_\xi$ on V_ξ such that the operators $\pi(\mathfrak{r},\xi)$ are equal to 1 on this multiplicity one isotype, and this trivializes the cocycle γ_ξ.

Proposition 6.12. *Let $\mathcal{H} = \mathcal{H}(\mathcal{R},q)$ be an affine Hecke algebra, and let $\xi = (P,\delta,t) \in \Xi_u$ be a standard tempered induction datum such that the central character of δ is positive (i.e. infinitesimally real). Then γ_ξ is trivial.*

Proof. In view of the above argument, it is sufficient to prove the existence of a multiplicity one W_0-type in $\pi(\xi)$. We thank Dan Ciubotaru for communicating to us that it can be shown that any irreducible representation of $\mathcal{H}$ *with positive central character* admits a multiplicity one W_0-type (see [6, Introduction, paragraph 1.3]). The proof of this fact is based on case-by-case verifications. Now consider $\xi = (P,\delta,t)$ with δ a discrete series with real central character. If $t \in T^P$ is positive and sufficiently generic, $\pi(\xi)$ is irreducible and has positive central character. By Ciubotaru's result mentioned above, this implies the existence of a multiplicity one W_0-type. As explained above, it follows that $\pi(\xi)$ has a multiplicity one W_0-type for all $t \in T^P$, hence in particular for all $t \in T^P_u$ as desired. $\square$

We do not know how to generalize this argument to general δ.

6.7.2. *Examples where γ_Δ is nontrivial.* In this subsection we present an example showing that γ_Δ is not trivial for $\mathcal{H}^D_n(\Lambda_{r,n})$ if $n > 8$. In the notation of the proof of Proposition 6.11, we write $l = n-1$ if n is odd, and $l = n-2$ is n is even, and put $l = 2m$ in both cases. We define $\lambda = (1)$ or $\lambda = (1,1)$ depending on n being odd or even. Recall that conjugate partition of λ is denoted by μ; hence we have $\mu = (1)$ in the first case and $\mu = (2)$ in the second case. By (6.6), (6.17), (6.20) and (6.21) we see that

$$\mathcal{W}_{P,P} = (W_0(B_{\mu_1}) \times \langle\kappa\rangle \times \langle\omega\rangle)^{\Sigma_{odd}} \subset \mathcal{W}^A_{P,P} \times \mathcal{W}^D_{P,P} \approx W_0(B_{\mu_1}) \times C_2^2 \tag{6.35}$$

where $\omega \in \mathfrak{W}^D_{P,P}$ and $\kappa \in K^D_P$ are as in the proof of Proposition 6.11, and where Σ_{odd} is the linear character which is equal to the product of the signs of a signed permutation in $W_0(B_{\mu_1})$, which is trivial in $\langle\kappa\rangle$, and which is nontrivial on $\langle\omega\rangle$. In particular the homomorphism $\pi^D : \mathcal{W}_{P,P} \to \mathcal{W}^D_{P,P}$ of (6.23) is surjective (even has a section), with

$$\mathcal{W}^D_{P,P} = \langle\kappa\rangle \times \langle\omega\rangle \approx C_2^2 \tag{6.36}$$

Equation (6.18) reduces in this case to

$$\mathcal{H}_P = \mathcal{H}^D_{2m}(\mathbb{Z}^{2m}) \tag{6.37}$$

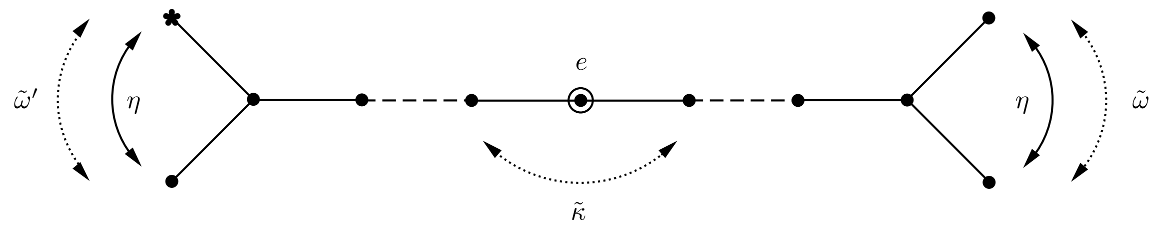

FIGURE 1. Spectral diagram of $\mathcal{H}_P$, with the action of $\tilde\omega$, $\tilde\omega'$ and $\tilde\kappa$.

and the action of $\mathcal{W}_{P,P}$ by automorphisms of $\mathcal{H}_P$ factors through the surjective projection π^D to an action (denoted by β) of $\mathcal{W}^D_{P,P}$ on $\mathcal{H}_P$ by automorphisms.

The spectral diagram (in the sense of [26, Definition 8.1]) of $\mathcal{H}_P$ is the affine Dynkin diagram of type D_{2m} equipped with the action of the unique nontrivial diagram automorphism η whose set of fixed points is the set of non-extremal vertices of the diagram. In Figure 1 this spectral diagram is displayed, with the action of η indicated by the solid arrows. In addition we have indicated in Figure 1 the middle vertex e (the encircled vertex) of the diagram, and the action of three diagram automorphisms $\tilde\kappa$, $\tilde\omega$ and $\tilde\omega'$ (by the dashed arrows). The group G of diagram automorphisms generated by $\tilde\kappa$ and $\tilde\omega$ is isomorphic to the dihedral group $\mathbb{D}_8$, and we have a projection $q : G \to \mathcal{W}^D_{P,P}$ with kernel $\langle\eta\rangle$ (the center of G). Observe that $\eta = \tilde\omega'\tilde\omega = \tilde\kappa\tilde\omega\tilde\kappa\tilde\omega$ in G. Recall that $T_P = \mathrm{Hom}(\mathbb{Z}^{2m}, \mathbb{C}^\times) = (\mathbb{C}^\times)^{2m}$, and denote by $\tilde T_P$ the double cover $\tilde T_P := \mathrm{Hom}(\Lambda^D_{2m}, \mathbb{C}^\times)$ of T_P. There is a canonical identification of the group of special diagram automorphisms $\langle\tilde\kappa\rangle \times \langle\eta\rangle$ with the group $\tilde K_P := (\Lambda^D_{r,2m})^\vee/(\Lambda^D_{2m})^\vee \approx C_2^2 \subset \tilde T^P$ of fixed points for natural action of $W_0(D_{2m})$ on $\tilde T_P$. We can extend the action of $W_0(D_{2m})$ on $\tilde T_P$ to $W_0(B_{2m}) = W_0(D_{2m}) \rtimes \langle\tilde\omega\rangle$. Then η is the unique nontrivial fixed point for this action of $W_0(B_{2m})$, while $\tilde\omega(\tilde\kappa) = \eta\tilde\kappa$. We have $T_P = \tilde T_P/\langle\eta\rangle$, and the action of $W_0(B_{2m})$ on T_P admits a unique nontrivial fixed point $\kappa = (-1, \dots, -1) = \tilde\kappa\langle\eta\rangle$. The group $G = \tilde K_P \rtimes \langle\tilde\omega\rangle$ acts naturally on $\mathcal{H}^D_{2m}(\Lambda_{2m})$ via an action $\tilde\beta$ defined as follows: $\tilde\beta(\tilde\omega)$ is the diagram automorphism arising from the automorphism of the root datum of $\mathcal{H}^D_{2m}(\Lambda_{2m})$ which $\tilde\omega$ induces, and $k \in \tilde K_P$ acts on the Bernstein basis of $\mathcal{H}^D_{2m}(\Lambda_{2m})$ by $\tilde\beta(k)(\theta_x N_w) = k(x)\theta_x N_w$. Since η is central in G, we see that $\mathcal{H}_P = (\mathcal{H}^D_{2m}(\Lambda_{2m}))^\eta$ is stable for the action of G via $\tilde\beta$. If we restrict the action of $\tilde\beta$ to the subalgebra $\mathcal{H}_P$, this restricted action descends to an action of $\mathcal{W}^D_{P,P}$ on $\mathcal{H}_P$ which coincides with the action β on $\mathcal{H}_P$ defined above.

Recall Lusztig's parameterization [16],[26, Theorem 8.7] of the discrete series representations of $\mathcal{H}_P = \mathcal{H}^D_{2m}(\mathbb{Z}^{2m})$. According to this result the discrete series representations of $\mathcal{H}_P$ whose central character $W_0(D_{2m})r \subset \mathrm{Hom}(\mathbb{Z}^{2m}, \mathbb{C}^\times)$ contains points with unitary part equal to $s(e) = (1, \dots, 1, -1, \dots, -1) \in T_P$ (with the same number m of 1's and -1's) correspond to discrete series representations of the extended graded affine Hecke algebra of the form

$$\mathbf{H}_e := (\mathbf{H}(D_m) \otimes \mathbf{H}(D_m)) \rtimes \langle\eta\rangle \tag{6.38}$$

where $\mathbf{H}(D_m)$ is shorthand for $\mathbf{H}(R_1, V, F_1, k)$ (in the notation of [26]), where R_1 is a root system of type D_m in V^*, with basis of simple roots F_1. The underlying based root system of the Hecke algebra $\mathbf{H}_e$ is the root system $R_{s(e),1}$ of type $D_m \times D_m$ obtained from the spectral diagram by deleting the vertex e. The group $G = \mathbb{D}_8$ of diagram automorphisms fixes e; this yields an action (denoted by α) of G on the algebra $\mathbf{H}_e$ by diagram automorphisms. Observe

that $\alpha(\eta)$ is inner on $\mathbf{H}_e$, hence α gives rise to a homomorphism of G to the group of outer automorphisms of $\mathbf{H}_e$ which factors through $\mathcal{W}^D_{P,P}$ via q.

To explain the above mentioned correspondence between the discrete series on both sides, recall from [26, proof of Theorem 7.1] that every discrete series representation δ of $\mathbf{H}(D_m)$ is fixed for twisting with the action of the unique nontrivial diagram automorphism $\tilde{\omega}$ of D_m, and can in fact be extended to $\mathbf{H}(B_m) = \mathbf{H}(D_m) \rtimes \langle\tilde{\omega}\rangle$ in precisely two ways, denoted by δ_+ and δ_-. In particular, the central character $W_0(D_m)\xi$ of δ is fixed for the action of $\tilde{\omega}$. If $\delta_+(\tilde{\omega}) = \Omega \in \mathrm{GL}(V_\delta)$ then $\delta_-(\tilde{\omega}) = -\Omega$. A discrete series representation of $\mathbf{H}_e$ is therefore of the form $\delta_{1,\epsilon_1} \otimes \delta_{2,\epsilon_2}$ (with $\epsilon_i = \pm$, and where δ_i is a discrete series representations of $\mathbf{H}(D_m)$) and has a central character of the form $c_V := ((W_0(D_m) \times W_0(D_m)) \rtimes \langle\eta\rangle)(\xi_1, \xi_2) = (W_0(D_m)(\xi_1), W_0(D_m)(\xi_2))$. Notice that the identity map defines an equivalence of $\mathbf{H}_e$-representations

$$\delta_{1,\epsilon_1} \otimes \delta_{2,\epsilon_2} \xrightarrow{\sim} \delta_{1,-\epsilon_1} \otimes \delta_{2,-\epsilon_2} \tag{6.39}$$

This representation corresponds (in the sense of [26, Theorem 8.7]) to a discrete series representation σ of $\mathcal{H}_P$ with central character $cc := W_0(D_{2m})(r_\sigma)$ with $r_\sigma := s(e)\exp(\xi_1, \xi_2)$. We note that $c := s(e)\exp c_V = s(e)(W_0(D_m)(\exp(\xi_1)), W_0(D_m)(\exp(\xi_2))) \subset cc$ is an equivalence class in the sense of [16, paragraph 8.1]. The correspondence discussed above is completely determined, using Lusztig's isomorphism [16, Section 8, Section 9]

$$\Phi : e_c(\bar{\mathcal{H}}_{P,cc})e_c \xrightarrow{\sim} \bar{\mathbf{H}}_{e,c_V}, \tag{6.40}$$

by the requirement that the action of $\mathbf{H}_e$ on $\sigma(e_c)(V_\sigma) \subset V_\sigma$ via Φ is equivalent to the representation $\delta_{1,\epsilon_1} \otimes \delta_{2,\epsilon_2}$. It is clear by the above that these representations of $\mathbf{H}_e$ are invariant for twisting by $\tilde{\omega}$, and that they are invariant for $\tilde{\kappa}$ if and only if $\delta_1 = \delta_2 := \delta$ (a discrete series representation of $\mathbf{H}(D_m)$). Write $\sigma(\delta)$ for the discrete series representation of $\mathcal{H}_P$ corresponding to $\delta_+ \otimes \delta_+$ and $\bar{\sigma}(\delta)$ for the one corresponding to $\delta_+ \otimes \delta_-$.

Proposition 6.13. *The discrete series representations of $\mathcal{H}_P$ of the form $\bar{\sigma}(\delta)$ are invariant for twisting by the group $\mathcal{W}^D_{P,P} = \langle\kappa\rangle \times \langle\omega\rangle \approx C_2^2$ of automorphisms of $\mathcal{H}_P$, and the corresponding factor set γ (see* [8, 8.32]*) is a nontrivial cocycle of $\mathcal{W}^D_{P,P}$.*

Proof. The invariance of $\bar{\sigma}(\delta)$ was discussed above. We trace the action β of $\mathcal{W}^D_{P,P}$ on $\bar{\mathcal{H}}_{P,cc}$ through Lusztig's isomorphism (6.40). But since the equivalence class c is ω-invariant but not κ invariant we are forced to work with the $\mathcal{W}^D_{P,P}$-invariant idempotent $e_c + e_{\kappa(c)}$ rather than e_c. Following Lusztig [16], we choose $w \in W_0(D_{2m})$ such that $\kappa(c) = wc$ and such that w has minimal length (or equivalently, such that $w(F_{s(e),1}) = F_{s(e),1}$). We have an algebra isomorphism

$$\begin{aligned} \Psi : (e_c + e_{wc})\bar{\mathcal{H}}_{P,cc}(e_c + e_{wc}) &\xrightarrow{\sim} \mathrm{Mat}_{2\times 2}(\bar{\mathbf{H}}_{e,c_V}) \\ X &\longrightarrow \begin{pmatrix} \Phi(e_c X e_c) & \Phi(e_c X e_{wc} i^0_w) \\ \Phi(i^0_{w^{-1}} e_{wc} X e_c) & \Phi(i^0_{w^{-1}} e_{wc} X e_{wc} i^0_w) \end{pmatrix} \end{aligned} \tag{6.41}$$

We transfer the action β of $\mathcal{W}^D_{P,P}$ on $(e_c + e_{wc})\bar{\mathcal{H}}_{P,cc}(e_c + e_{wc})$ to the matrix algebra on the right hand side via Ψ; we shall denote the resulting action of $\mathcal{W}^D_{P,P}$ by μ. We use the isomorphism $\mathrm{Mat}_{2\times 2}(\bar{\mathbf{H}}_{e,c_V}) = \mathrm{Mat}_{2\times 2}(\mathbb{C}) \otimes \bar{\mathbf{H}}_{e,c_V}$, and write I for the identity automorphism of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$. Using the results of [16, Section 8] it is not difficult to show that

$$\mu(\omega) := \Psi \circ \beta(\omega) \circ \Psi^{-1} = C_{\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}} \circ (I \otimes \alpha(\tilde{\omega})) \tag{6.42}$$

where, for an invertible matrix M, C_M denotes the inner automorphism of conjugation with M. Similarly, we see that

$$\mu(\kappa) := \Psi \circ \beta(\kappa) \circ \Psi^{-1} = C_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \circ (I \otimes \alpha(\tilde{\kappa})) \tag{6.43}$$

Observe the relation $(\mu(\kappa)\mu(\omega))^2 = \mathrm{Id}$ (use $(\alpha(\tilde{\kappa})\alpha(\tilde{\omega}))^2 = \alpha(\eta)$, the inner automorphism of conjugation by η on $\mathbf{H}_{e,c_V}$), showing that we indeed defined a representation of $\mathcal{W}^D_{P,P}$, and not just of G. Recall that $\bar{\sigma}(\delta)$ has the defining property that $\mathbf{H}_e$ acts via Φ on $\bar{\sigma}(\delta)(e_c)(V_{\bar{\sigma}(\delta)}) \approx V_\delta \otimes V_\delta$ according to $\delta_+ \otimes \delta_-$. It follows that $\mathrm{Mat}_{2\times 2}(\bar{\mathbf{H}}_{e,c_V}) = \mathrm{Mat}_{2\times 2}(\mathbb{C}) \otimes \bar{\mathbf{H}}_{e,c_V}$ acts via Ψ on $V_{c,wc} := \bar{\sigma}(\delta)(e_c + e_{wc})(V_{\bar{\sigma}(\delta)}) \approx \mathbb{C}^2 \otimes (V_\delta \otimes V_\delta)$ by $\mathrm{id} \otimes (\delta_+ \otimes \delta_-)$ (here id denotes the defining action of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ on $\mathbb{C}^2$). We write elements of $\mathbb{C}^2 \otimes (V_\delta \otimes V_\delta)$ as a column vector of size 2 with entries in $V_\delta \otimes V_\delta$, so that the action of $\mathrm{Mat}_{2\times 2}(\bar{\mathbf{H}}_{e,c_V})$ can be written as matrix multiplication where the matrix entries act on $V_\delta \otimes V_\delta$ via $\delta_+ \otimes \delta_-$.

It follows straight from the definition of the correspondence that the factor set γ for $\mathcal{W}^D_{P,P}$ defined the by module $\bar{\sigma}(\delta)$ via the action β is equal to the factor set defined by the module $V_{c,wc}$ via the action μ. The following linear involutions $M(\kappa), M(\omega)$ on $V_{c,wc}$ define intertwining operators from $V_{c,wc}$ to its twists by $\mu(\kappa)$ and $\mu(\omega)$ respectively:

$$M(\kappa)\begin{pmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{pmatrix} = \begin{pmatrix} v_2 \otimes v_1 \\ u_2 \otimes u_1 \end{pmatrix};\ M(\omega)\begin{pmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{pmatrix} = \begin{pmatrix} -u_1 \otimes \Omega(u_2) \\ \Omega(v_1) \otimes v_2 \end{pmatrix} \tag{6.44}$$

We see that $(M(\kappa)M(\omega))^2 = -\mathrm{Id}$, proving that M lifts to a linear representation of the Schur extension $G = \mathbb{D}_8$ of $\mathcal{W}^D_{P,P}$ in which η acts by $-\mathrm{Id}$. In particular, γ is nontrivial. □

Corollary 6.14. *The 2-cocycle γ_Δ of $\mathcal{W}_\Delta$ for $\mathcal{H}^D_n(\Lambda_{r,n})$ is nontrivial if $n > 8$.*

Proof. By definition (cf. paragraph 3.2.3) the pullback of γ_Δ to $\mathcal{W}_{P,P} \approx (\mathcal{W}_\Delta)_{(P,\bar{\sigma}(\delta)),(P,\bar{\sigma}(\delta))}$ is equal to the pullback $(\pi^D)^*(\gamma)$ of the factor set γ via $\pi^D : \mathcal{W}_{P,P} \to \mathcal{W}^D_{P,P}$. Using (6.35) it is easy to see that π^D has a section $s : \mathcal{W}^D_{P,P} \to \mathcal{W}_{P,P}$, implying that $(\pi^D)^*$ is injective on cohomology classes by contravariant functoriality of $H^2(?, \mathbb{C}^\times)$. By the above Proposition we conclude that $(\pi^D)^*(\gamma)$ is a nontrivial 2-cocycle, implying that γ_Δ is nontrivial. □

7. Appendix: The Weyl groupoid

In this paragraph and in the next we recall some well known facts about Weyl groups and standard parabolic subgroups of Weyl groups. These results are essentially due to Langlands, and the basic references for this material are [5], [19].

Let $\mathfrak{a} = \mathrm{Lie}(T_{\mathrm{rs}})$ be the finite dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} Y$. Then $R_0 \subset \mathfrak{a}^*$ is a reduced, integral root system. Recall however that we do not assume that $\mathbb{R}R_0 = \mathfrak{a}^*$.

The Weyl group W_0 of R_0 acts naturally as a real reflection group on $\mathfrak{a}$. The set of simple reflections in W_0 corresponding to the basis of simple roots F_0 is denoted by S_0.

A parabolic subgroup of W_0 is the isotropy subgroup of an element of $\mathfrak{a}$. A standard parabolic subgroup of W_0 is a subgroup $W_P \subset W_0$ which is generated by the set of simple reflections corresponding to a subset $P \subset F_0$. Clearly every parabolic subgroup is conjugate to a standard parabolic subgroup.

Let us denote by $\mathcal{P}$ the power set of F_0. Given $P, Q \in \mathcal{P}$ we denote $\mathfrak{W}_{P,Q} := \{w \in W_0 \mid w(P) = Q\} \subset W_0$.

Definition 7.1. *The Weyl groupoid $\mathfrak{W}$ is the finite groupoid whose set of objects is $\mathcal{P}$ and* $\mathrm{Hom}_{\mathfrak{W}}(P,Q) := \mathfrak{W}_{P,Q}$

For a standard parabolic subgroup W_P we have a distinguished set W^P of left coset representatives for W_P, characterized by $W^P := \{w \in W_0 \mid w(P) \subset R_{0,+}\}$. We denote by w_0 the longest element of W_0, and by w_P the longest element of W_P. Then the longest element in W^P is equal to $w^P = w_0 w_P$. Observe that $\bar{P} := w^P(P) \in \mathcal{P}$, so that we always have $w^P \in \mathfrak{W}_{P,\bar{P}}$. The element $\bar{P} \in \mathcal{P}$ is called the conjugate of P.

If $P, Q \in \mathcal{P}$ and $\mathfrak{W}_{P,Q} \neq \emptyset$ then P, Q are called associates. In particular, for every $P \in \mathcal{P}$ the conjugates P and $\bar{P}$ are associates.

Given $P \in \mathcal{P}$ we put $\mathfrak{a}^P = \{x \in \mathfrak{a} \mid \alpha(x) = 0 \forall \alpha \in P\}$. Consider the set $\mathfrak{S} := \{(P, x) \mid P \in \mathcal{P},\ x \in \mathfrak{a}^P\}$. Then $\mathfrak{S}$ is a collection of real vector spaces which is naturally fibred over $\mathcal{P}$. The set $\mathfrak{S}$ carries a natural action of $\mathfrak{W}$ defined by $w(P, x) = (Q, wx)$ if $w \in \mathfrak{W}_{P,Q}$.

7.0.3. *Chamber system of* $\mathfrak{W}$. We denote by $\mathfrak{a}^+$ the positive Weyl chamber in $\mathfrak{a}$. Every face of $\overline{\mathfrak{a}^+}$ is of the from $\overline{\mathfrak{a}^+} \cap \mathfrak{a}^P$ for a unique $P \in \mathcal{P}$, and this sets up natural bijection between the facets of $\overline{\mathfrak{a}^+}$ and $\mathcal{P}$.

The subset of R_0 consisting of the roots of W_P is denoted by R_P, thus $R_P = R_0 \cap \mathbb{R}P$. We choose the set of positive roots $R_{P,+}$ in R_P corresponding to the basis P of R_P.

We adopt the notation (P, α) to denote the restriction of $\alpha \in R_0 \backslash R_P$ to $\mathfrak{a}^P$. We write $R^P \subset \mathfrak{a}^{P,*} \backslash \{0\}$ for the set of restrictions (P, α) of roots $\alpha \in R_0 \backslash R_P$ which are in addition primitive in the sense that if $\beta \in R_0 \backslash R_P$ and $(P, \alpha) \in R^P$ such that (P, α) and (P, β) are proportional, then $(P, \beta) = c(P, \alpha)$ with $c \in \mathbb{Z}$. We write R^P_+ for the primitive restrictions corresponding to the positive roots $\alpha \in R_{0,+} \backslash R_{P,+}$. An element (P, α) is called *simple* if (P, α) is indecomposable in $\mathbb{Z}_+ R^P_+$. This is equivalent to saying that (P, α) is the restriction of an element of $F_0 \backslash P$.

To each $(P, \alpha) \in R^P$ we associate a hyperplane $(P, H_\alpha) = \mathrm{Ker}(P, \alpha) \subset \mathfrak{a}^P$. The hyperplanes (P, H_α) are called the *walls* in $\mathfrak{a}^P$.

A chamber of $\mathfrak{W}$ in $\mathfrak{S}$ is a pair (P, C) with $P \in \mathcal{P}$, and $C \subset \mathfrak{a}^P$ a connected component of the complement of the collection of walls in $\mathfrak{a}^P$. The collection of chambers is denoted by $C(\mathfrak{W}, \mathfrak{a}, F_0)$. This is a finite set, which has a natural fibration $(P, C) \to P$ over the set $\mathcal{P}$. The action of $\mathfrak{W}$ on $\mathfrak{S}$ maps chambers to chambers, and thus induces a natural action of the groupoid $\mathfrak{W}$ on $C(\mathfrak{W}, \mathfrak{a}, F_0)$.

The set R^P_+ determines a distinguished chamber $(P, \mathfrak{a}^{P,+})$ of $\mathfrak{a}^P$, defined by $\mathfrak{a}^{P,+} = \{x \in \mathfrak{a}^P \mid \alpha(x) > 0\ \forall \alpha \in R^P_+\}$. Observe that the chambers are simplicial cones, and that $\overline{(P, \mathfrak{a}^{P,+})}$ is the face of $\overline{\mathfrak{a}^+}$ corresponding to P.

An (irredundant) gallery of length n in $\mathfrak{a}^P$ is a sequence $C_0, C_1, \dots, C_n$ of chambers contained in $\mathfrak{a}^P$ such that each pair C_{i-1}, C_i $(i = 1, \dots, n)$ consists of distinct chambers which share a common face. A minimal gallery is a gallery of shortest length between its end points. The distance between two chambers is the length of a minimal gallery between them.

Given a chamber (P, C), we define its height $\mathrm{ht}(P, C)$ to be the number of walls of $\mathfrak{a}^P$ separating $(P, \mathfrak{a}^{P,+})$ and C. Thus $\mathrm{ht}(P, C)$ is equal to the distance between $(P, \mathfrak{a}^{P,+})$ and C.

7.0.4. *Elementary conjugations.* The faces of $\mathfrak{a}^{P,+}$ are of the form $\mathfrak{a}^{Q,+} = \overline{\mathfrak{a}^{P,+}} \cap (P, H_\alpha)$, where $Q \in \mathcal{P}$ is such that $P \subset Q$ and $Q \backslash P = \{\alpha\}$. Thus the faces of $\mathfrak{a}^{P,+}$ are in bijective correspondence with the $Q \in \mathcal{P}$ containing P as a maximal proper subset. Given $Q \in \mathcal{P}$ containing P as a maximal proper subset we define an element $\mathfrak{s}^P_Q \in \mathfrak{W}_{P,P'}$ by $\mathfrak{s}^P_Q = w_Q w_P$.

Here $P' = \mathfrak{s}_Q^P(P) \subset Q$ is the conjugate of P in Q. Notice that $\mathfrak{s}_Q^{P'} = (\mathfrak{s}_Q^P)^{-1}$. In particular, $P \subset Q$ is self-opposed (in the terminology of [4, Section 10.4], i.e. $P \subset Q$ is its own conjugate as a maximal standard parabolic subsystem of Q) iff $\mathfrak{s}_Q^P$ is an involution. The following result is well known (see [5],[19]).

Theorem 7.2. (i) *The action of $\mathfrak{W}$ on the set $C(\mathfrak{W}, \mathfrak{a}, F_0)$ of chambers of $\mathfrak{W}$ is free, and every $\mathfrak{W}$-orbit in $C(\mathfrak{W}, \mathfrak{a}, F_0)$ contains a unique positive chamber $(P, \mathfrak{a}^{P,+})$ (with $P \in \mathcal{P}$).*

(ii) *Every element $(P, x) \in \mathfrak{S}$ is $\mathfrak{W}$-conjugate to a $(Q, y) \in \mathfrak{S}$ with $y \in \overline{\mathfrak{a}^{Q,+}}$.*

(iii) *If (P, C_1) and (P, C_2) are distinct neighboring chambers then $C_1 = w_1(\mathfrak{a}^{P_1,+})$, $C_2 = w_2(\mathfrak{a}^{P_2,+})$ and $w_1^{-1}w_2$ is the elementary conjugation $\mathfrak{s}_Q^{P_2}$ in $\mathfrak{W}$ with respect to a uniquely determined $Q \in \mathcal{P}$ which contains both P_2 and P_1 as maximal proper subsets.*

Corollary 7.3. (i) *Every $w \in \mathfrak{W}_{P,Q}$ can be written as a product of elementary conjugations in $\mathfrak{W}$.*

(ii) *The minimal length of a word consisting of elementary conjugations representing $w \in \mathfrak{W}_{P,Q}$ is equal to the height of $(Q, w(\mathfrak{a}^{P,+}))$.*

(iii) *The reduced expressions for $w \in \mathfrak{W}_{P,Q}$ as a product of elementary conjugations correspond bijectively to the minimal galleries in $\mathfrak{a}^Q$ from $\mathfrak{a}^{Q,+}$ to $w(\mathfrak{a}^{P,+})$. If $\mathfrak{a}^{Q,+} = C_1, C_2, \ldots, C_n = w(\mathfrak{a}^{P,+})$ is a minimal gallery with $C_i = w_i(\mathfrak{a}^{P_i,+})$ and we put $x_i = w_{i-1}^{-1}w_i \in \mathfrak{W}_{P_i,P_{i-1}}$, then $w = x_1 \cdot \cdots \cdot x_n$ is the corresponding reduced expression for w.*

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